Concentration.

Sunday, March 30, 2025 5:56 PM

We have seen Chebyshev's inequality:

$$\mathbb{P}[\,|X-\mathbb{E}X|>\beta\,] \le \frac{\operatorname{Var}X}{\beta^2}$$

- Today: more concentration inequalities
  - Chernoff - Cramér bound $\Rightarrow$ large-deviation
  - Bernstein inequality / Hoeffding inequality (sum of independent)
  - Azuma-Hoeffding inequality / bounded difference estimates (martingale)

Applications:
- norm bound of random matrices
- first-passage percolation
- chromatic number of graph.

- Chernoff - Cramér : $\mathbb{P}[X\ge\beta]\le e^{-s\beta}\,\underline{\mathbb{E}[e^{sX}]}$, $\forall s\ge 0$

  proof: Markov inequality. moment generating function (MGF) $= \sum_{k=0}^{\infty}\frac{s^k\,\mathbb{E}[X^k]}{k!}$

- Useful for sum of independent:

  MGF factors: $\mathbb{E}\left[e^{s\sum_{i=1}^m X_i}\right]=\prod_{i=1}^m \mathbb{E}[e^{sX_i}]$

- Example : Large deviation

$\Lambda(s)=\log \mathbb{E}_\mu[e^{sX}]$ cumulant-generating function. $\left[\begin{array}{l}\Lambda \text{ convex}\\ \Lambda_* \text{ convex}\end{array}\right]$

Legendre transform: $\Lambda_*(x)=\sup_s\, sx-\Lambda(s)$

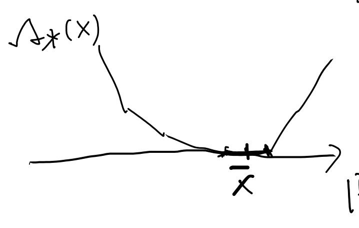
$\hookrightarrow$ ≥ 0 since $\Lambda(0)=0$ $\quad(\bar{X}=\mathbb{E}X)$

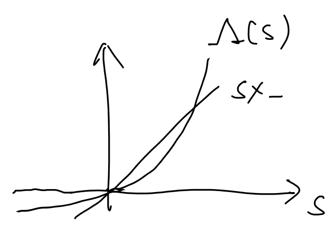
$\Lambda_*(\bar{X})=0$, since $\Lambda(s)=\log \mathbb{E}_\mu[e^{sX}]\ge \mathbb{E}_\mu[\log e^{sX}]=s\bar{X}$

$A_m=\frac{1}{m}(X_1+\cdots+X_m)$, iid $\sim\mu$

(Cramér) $-\frac{1}{m}\log(\mathbb{P}[A_m\in I])\to \inf_{x\in I}\Lambda_*(x)$ as $m\to\infty$

<u>proof</u>. Upper bound: $\mathbb{P}[A_m\in I]\le \exp(-m\inf_{x\in I}\Lambda_*(x))$





If $\bar{x}\in I$, obvious

Assume $\inf I = x_- > \bar{x}$

$\mathbb{P}[A_m\ge x_-]\le \exp(-smx_- + m\Lambda(s))$, $\forall s\ge 0$

$\le \exp(-m\Lambda_*(x_-))$ $\quad$ [Here $sx_- - \Lambda(s)$ maximized when $s\ge 0$, since $\Lambda'(0)=\mathbb{E}X=\bar{x}<x_-$)]

Lower bound: $\mathbb{P}[A_m\in I]\gtrsim \exp(-m\Lambda_*(x))$ for any $x\in I$.

change of measure: $\frac{\tilde{\mu}}{\mu}(x)=\exp(sx-\Lambda(s))$

$$\mathbb{P}[A_m\in(x-\varepsilon,x+\varepsilon)]\approx \mathbb{E}\left[\exp(\underset{\sim\sim\sim}{-m\tilde{A}_m s+m\Lambda(s)})\,\mathbb{1}[\tilde{A}_m\in(x-\varepsilon,x+\varepsilon)]\right]$$

$$\approx \exp\left(-m\,(sx-\Lambda(s))\right)\,\mathbb{P}[\tilde{A}_m\in(x-\varepsilon,x+\varepsilon)]$$

choose s such that $\mathbb{P}[\tilde{A}_m\in(x-\varepsilon,x+\varepsilon)]\to 1$ $\quad(\mathbb{E}_{\tilde{\mu}}X=x)$

- Some concentration inequalities

Take $X_1,\dots,X_n$ independent. with $\mu_i=\mathbb{E}X_i$, $\sigma_i^2=\operatorname{Var}(X_i)$ and $|X_i-\mu_i|\le c_i$ $\quad c=\max c_i$

For $T=X_1+\cdots+X_n$. $\Rightarrow \mathbb{P}[T-\mathbb{E}T>\beta] < \begin{cases}\exp\left(-\beta^2/4\sum_{i=1}^n\sigma_i^2\right) & 0\le\beta\le\frac{\sum\sigma_i^2}{c}\\ \exp(-\beta/4c) & \beta>\frac{\sum\sigma_i^2}{c}\end{cases}$ $\Big/$ $\exp\left(-\frac{\beta^2}{2\sum_{i=1}^n\sigma_i^2+2c\beta/3}\right)$

(Bernstein)

$$\mathbb{P}[T-\mathbb{E}T>\beta] < \exp\left(-2\beta^2\Big/\sum_{i=1}^n c_i^2\right)$$

(Hoeffding) $\quad$ (Both for sub-Gaussian as well)

<u>proof</u> Using Chernoff-Cramér; need: compute MGF

(Bernstein) $\mathbb{E}\exp(s(X_i-\mu_i))$

$$=\sum_{k=0}^{\infty}\frac{s^k\,\mathbb{E}(X_i-\mu_i)^k}{k!}\le 1+\sum_{k=2}^{\infty}\frac{s^k}{k!}\sigma_i^2\cdot c^{k-2}\le 1+\frac{s^2\sigma_i^2}{2}+\frac{s^3\sigma_i^2}{6}\sum_{k=3}^{\infty}(sc)^{k-2}=1+\frac{s^2\sigma_i^2}{2}\left(1+\frac{1}{3}\frac{sc}{1-sc}\right)\le 1+s^2\sigma_i^2\le\exp(s^2\sigma_i^2)$$

$(\mathbb{E}(X_i-\mu_i)^k\le\mathbb{E}(X_i-\mu_i)^2\cdot c^{k-2})$ $\quad\downarrow$ when $s\le\frac{1}{2c}$

(Hoeffding) $\mathbb{E}\exp(s(X_i-\mu_i))\le\frac{1}{2}(e^{c_is}+e^{-c_is})\le e^{2c_i^2s^2}$

- Operator norm for random matrices

$X=(X_{ij})$ $m\times n$ random matrix,

$$\|X\|_2=\sup_{\substack{\|u\|_2=\|v\|_2=1\\ u\in\mathbb{R}^m,\ v\in\mathbb{R}^n}}u^TXv$$

Suppose X_{ij} iid rademacher $(\mathbb{P}[X_{ij}=1]=\mathbb{P}[X_{ij}=-1]=\frac{1}{2})$

<u>Claim</u>. $\mathbb{P}[\|X\|_2\ge C(\sqrt{m}+\sqrt{n}+t)]<e^{-t^2}$

($C>0$ is some constant)

<u>proof</u>. ① Bound u^TXv for fixed u,v

$u^TXv=\sum_{i=1}^m\sum_{j=1}^n u_iv_jX_{ij}$

$$\mathbb{P}[u^TXv\ge C(\sqrt{m}+\sqrt{n}+t)]\underset{\text{(Hoeffding)}}{<}\exp\left(-\frac{2C^2(\sqrt{m}+\sqrt{n}+t)^2}{\sum_{i=1}^m\sum_{j=1}^n u_i^2v_j^2}\right)=\exp(-2C^2(\sqrt{m}+\sqrt{n}+t)^2)\le\exp(-2C^2(m+n+t^2))$$

② For $\mathbb{S}^{m-1}$

(i.e. $\{u\in\mathbb{R}^m:\|u\|_2=1\}$) $\quad$ ($\frac{1}{4}$-net)

$\exists\ P_m\subseteq\mathbb{S}^{m-1}$, $|P_m|=12^m$, $\bigcup_{x\in P_m}B_{\frac14}(x)\supseteq\mathbb{S}^{m-1}$

$\Rightarrow \mathbb{P}\left[\max_{\substack{u\in P_m\\ v\in P_n}}u^TXv\ge C(\sqrt{m}+\sqrt{n}+t)\right]<e^{-t^2}$ (union bound)

③ $\forall\, u\in P_m$, $v\in P_n$, $\tilde{u}\in\mathbb{S}^{m-1}$, $\tilde{v}\in\mathbb{S}^{n-1}$, $\|u-\tilde{u}\|_2\le\frac14$, $\|v-\tilde{v}\|_2\le\frac14$,

$$|u^TXv-\tilde{u}^TX\tilde{v}|\le|(u^T-\tilde{u}^T)Xv|+|\tilde{u}^TX(v-\tilde{v})|$$
$$\le\frac14\|X\|_2+\frac14\|X\|_2=\frac12\|X\|_2$$

$\Rightarrow$ $\tilde{u}^TX\tilde{v}\le\frac12\|X\|_2+C(\sqrt{m}+\sqrt{n}+t)$, with prob $>1-e^{-t^2}$

$\Rightarrow$ $\|X\|_2\le 2C(\sqrt{m}+\sqrt{n}+t)$, with prob $>1-e^{-t^2}$

Martingale concentration

Recall: Martingale $Z_1,Z_2,\dots,Z_n,\dots$

s.t. $\mathbb{E}|Z_n|<\infty$

$\mathbb{E}[Z_{n+1}\mid Z_1,\dots,Z_n]=Z_n$ $\quad$ (A_i,B_i can depended on $X_1,\dots,X_{i-1}$) (but C_i is constant)

Azuma / Azuma-Hoeffding ineq: $A_i\le Z_i-Z_{i-1}\le B_i$, $\quad B_i-A_i\le C_i$

$$\mathbb{P}\left[\max_{0\le i\le n}Z_i-Z_0\ge\beta\right]\le\exp\left(-\frac{2\beta^2}{\sum_{i=1}^nC_i^2}\right)$$

<u>proof</u> $\mathbb{P}\left[\max_{0\le i\le n}Z_i-Z_0\ge\beta\right]$

$\le\mathbb{P}\left[\max_{0\le i\le n}\exp(s(Z_i-Z_0))\ge e^{s\beta}\right]$

$\le\mathbb{E}[\exp(s(Z_\tau-Z_0))]\,e^{-s\beta}$

$\tau:=\min\{t:Z_t-Z_0\ge\beta\}$ or n $\quad$ (stopping time)

$\le\mathbb{E}[\exp(s(Z_n-Z_0))]\,e^{-s\beta}=\mathbb{E}\left[\exp\left(s\sum_{i=1}^n(Z_i-Z_{i-1})\right)\right]e^{-s\beta}$

Then Chernoff - Cramér.

- Bounded difference estimates.

$X_1,X_2,\dots,X_n$ independent

each $X_i\in\Omega_i$

$f:\Omega_1\times\cdots\times\Omega_n\to\mathbb{R}$

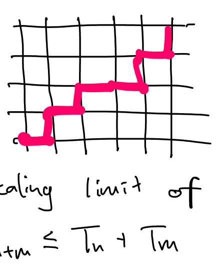
$$D_i=\max_{\substack{x_1\in\Omega_1,\dots,x_n\in\Omega_n\\ \tilde{x}_i\in\Omega_i}}|f(x_1,\dots,x_i,\dots,x_n)-f(x_1,\dots,\tilde{x}_i,\dots,x_n)|$$

Let $Z_i=\mathbb{E}[f(X_1,\dots,X_n)\mid X_1,\dots,X_i]$ $\quad$ (McDiarmid's ineq)

$\Rightarrow A_i\le Z_i-Z_{i-1}\le B_i$, $\quad B_i-A_i\le D_i$

$\Rightarrow \mathbb{P}[f(X_1,\dots,X_n)-\mathbb{E}f(X_1,\dots,X_n)\ge\beta]\le\exp\left(-\frac{2\beta^2}{D_1^2+\cdots+D_n^2}\right)$

- Example: $\frac{\text{First}}{\text{Last}}$- Passage Percolation



$w_e\sim\mathrm{Unif}[0,1]$, $\forall e\in E$ (edge weights)

$T_n=\min_\gamma\sum_\gamma w_e$

γ: up-right path from $(1,1)$ to (n,n)

scaling limit of T_n as $n\to\infty$?

$T_{n+m}\le T_n+T_m$ $\quad$ (sub-additivity)

$\Rightarrow \mathbb{E}T_{n+m}\le\mathbb{E}T_n+\mathbb{E}T_m$, $\quad\frac1n\mathbb{E}T_n$ converges as $n\to\infty$

(Exercise: $\lim_{n\to\infty}\frac1n\mathbb{E}T_n>0$, by large deviation)

By bounded difference estimates,

(X_i: w_e for e between $x+y=i-1$ and $x+y=i$)

$\mathbb{P}[T_n-\mathbb{E}T_n\ge\beta]\le\exp\left(-\frac{\beta^2}{4n}\right)$ $\quad(D_1=D_2=\cdots=D_{2n}=1)$

(In particular, $\operatorname{var}(T_n)$ is $O(n)$)

(KPZ universality conjecture: $\operatorname{var}(T_n)$ is of order $O(n^{\frac23})$)

(state of art: $\underset{\text{(Newman-Piza)}}{c\log(n)<}\operatorname{Var}(T_n)\underset{\text{(Benjamini-Kalai-Schramm)}}{<\frac{Cn}{\log(n)}}$)

Example. • Pattern matching

$X_1,X_2,\dots,X_n$, iid, each uniform from $\{1,\dots,s\}$

For $a=(a_1,\dots,a_k)\in\{1,\dots,s\}^k$

N_n: # i such that $(X_i,\dots,X_{i+k-1})=(a_1,\dots,a_k)$

$\mathbb{E}N_n=(n-k+1)s^{-k}$

$$\mathbb{P}[|N_n-(n-k+1)s^{-k}|\ge bk\sqrt{n}]\le 2\exp\left(-\frac{2b^2k^2n}{k^2n}\right)\le 2e^{-2b^2}$$

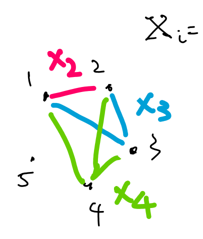
$\downarrow$ $D_i\le k$

- Chromatic number

Take Erdös-Rényi graph $G(n,p)$

χ: minimum number of colors to properly color $G(n,p)$

$X_i=\{\mathbb{1}[(i,j)\in G(n,p)]:1\le j\le i\}$



Claim: change X_i alters χ by ≤ 1

$(D_i\le 1)$

$$\Rightarrow \mathbb{P}[|\chi-\mathbb{E}\chi|\ge b\sqrt{n-1}]\le 2\exp\left(-\frac{2b^2(n-1)}{n-1}\right)=2e^{-2b^2}$$