Coupling & FKG.

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Today: Stochastic domination, Strassen's theorem

correlation inequalities: FKG; proof via Glauber dynamics

- Coupling: Let μ, ν be prob measures on Ω; a coupling of μ, ν is a probability measure γ on $\Omega \times \Omega$ s.t. $\gamma(A \times \Omega) = \mu(A)$, $\gamma(\Omega \times A) = \nu(A)$, $\forall A \subseteq \Omega$ measurable

Example. Poissonization of independent Bernoulli

$X_1 \sim Ber(p_1)$, $X_2 \sim Ber(p_2)$, $\cdots$, $X_n \sim Ber(p_n)$, independently

$S = X_1 + \cdots + X_n$

Then S can be coupled to $Z \sim Poi(\lambda)$, $\lambda = \lambda_1 + \cdots + \lambda_n$, $\lambda_i = -\log(1-p_i)$

s.t. $\mathbb{P}[S \neq Z] \leq \frac{1}{2} \sum_{i=1}^{n} \lambda_i^2$ ($\to 0$ if fix λ, $\max_{1 \leq i \leq n} \lambda_i \to 0$)

Proof $W_i = Poi(\lambda_i)$; $Z = \sum_{i=1}^{n} W_i$

$e^{-\lambda_i} = 1 - p_i = \mathbb{P}[W_i = 0] = \mathbb{P}[X_i = 0]$

$\Rightarrow$ can couple W_i, X_i s.t. $\mathbb{P}[W_i \neq X_i] = \mathbb{P}[W_i \geq 2] \leq \frac{\lambda_i^2}{2}$

- Stochastic domination:

Real variables: X stochastically dominates Y, if $\mathbb{P}[X > a] \geq \mathbb{P}[Y > a]$, for any $a \in \mathbb{R}$

example: $Poi(\lambda)$ s.d. $Ber(p)$, $\lambda \geq -\log(1-p)$

Thm X s.d. Y, iff there is a coupling $(\tilde{X}, \tilde{Y})$ of X, Y, s.t. $\mathbb{P}[\tilde{X} \geq \tilde{Y}] = 1$

proof. ① If such a coupling exists, for any $a \in \mathbb{R}$

$\mathbb{P}[Y > a] = \mathbb{P}[\tilde{Y} > a] = \mathbb{P}[\tilde{X} \geq \tilde{Y} > a] \leq \mathbb{P}[\tilde{X} > a] = \mathbb{P}[X > a]$

② If X s.d. Y: define $f_X(a) = \mathbb{P}[X \leq a]$, $f_Y(a) = \mathbb{P}[Y \leq a]$ $\Rightarrow$ $f_X(a) \leq f_Y(a)$

Take $U \sim unif[0,1]$. Let $\tilde{X} = f_X^{-1}(U)$, $\tilde{Y} = f_Y^{-1}(U)$

$\tilde{X} \overset{d}{=} X$, $\tilde{Y} \overset{d}{=} Y$, $\mathbb{P}[\tilde{X} \geq \tilde{Y}] = \mathbb{P}[f_X^{-1}(U) \geq f_Y^{-1}(U)] = 1$

Cor. For X s.d Y, and $f: \mathbb{R} \to \mathbb{R}$ non-decreasing. $f(X)$ s.d. $f(Y)$

And if $\mathbb{E}|f(X)|, \mathbb{E}|f(Y)| < \infty$, $\mathbb{E} f(X) \geq \mathbb{E} f(Y)$

Partially ordered sets. (POSET)

$(\Omega, \leq)$: for any $x, y, z \in \Omega$

① $x \leq x$

② if $x \leq y$, $y \leq x \Rightarrow x = y$

③ if $x \leq y$, $y \leq z \Rightarrow x \leq z$.

example: $\mathbb{R}^d$: $(x_1, \cdots, x_d) \leq (y_1, \cdots, y_d)$ iff $x_i \leq y_i$ for each $i \in \{1, \cdots, d\}$

Increasing set: $A \subseteq \Omega$ s.t. $x \in A$, $x \leq y \Rightarrow y \in A$

Increasing function: $f: \Omega \to \mathbb{R}$ s.t. $x \leq y \Rightarrow f(x) \leq f(y)$

Stochastic domination: X, Y random in Ω, X s.d. Y if $\mathbb{P}[X \in A] \geq \mathbb{P}[Y \in A]$ $\forall$ increasing A

(Strassen's thm) X, Y be random variables in a finite poset $(\Omega, \leq)$

X s.d. Y iff there is a coupling $(\tilde{X}, \tilde{Y})$ s.t. $\tilde{X} \geq \tilde{Y}$.

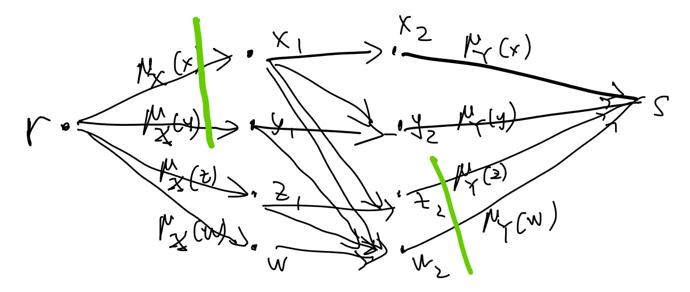
proof ① If $\exists$ coupling, $\mathbb{P}[Y \in A] \leq \mathbb{P}(X \in A]$ $\forall$ increasing A

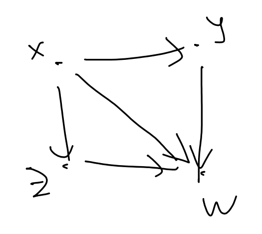
② If X s.d. Y, construct coupling.

i.e. give μ_X, μ_Y on Ω. want $\nu: \Omega \times \Omega \to \mathbb{R}_{\geq 0}$

s.t. $\nu(x, y) > 0$ only for $x \geq y$; and $\sum_y \nu(x, y) = \mu_X(x)$, $\sum_x \nu(x, y) = \mu_Y(y)$

Idea: max-flow = min-cut theorem $\Omega = \{x, y, z, w\}$





$x_1 \to y_2$ iff $x \geq y$, $cap(x_1 \to y_2) = \infty$

$cap(r \to x_1) = \mu_X(x_1)$, $cap(x_2 \to s) = \mu_Y(x_2)$

If max flow $= 1$ $\Rightarrow$ $\nu(x, y) = flow(x \to y)$

Theorem: max flow = min cut

$R: \{x : (x_2 \to s)$ is cut$\}$

$L: \{x : (r \to x_1)$ is cut$\} \supseteq L^* : \{x : x \geq y, \exists y \in \Omega \setminus R\}$

- L^* is increasing
- $L^* \supseteq \Omega \setminus R$

cutsize $= \mu_X(L) + \mu_Y(R) \geq \mu_X(L^*) + \mu_Y(\Omega \setminus L^*) = \mu_X(L^*) + 1 - \mu_Y(L^*) \geq 1$.

Cor. X, Y random variables in Ω, X s.d. Y $\Rightarrow$ $f(X)$ s.d. $f(Y)$; if $\mathbb{E}|f(X)|, \mathbb{E}|f(Y)| < \infty$, $\Rightarrow \mathbb{E} f(X) \geq \mathbb{E} f(Y)$

$f: \Omega \to \mathbb{R}$ increasing function

(Positive association) A measure μ on a poset Ω has P.a. if for any two increasing events A, B, $\mu(A \cap B) \geq \mu(A)\mu(B)$

Claim: for any increasing functions $f, g: \Omega \to \mathbb{R}$, with $\mathbb{E}_\mu|f|$, $\mathbb{E}_\mu|g|$, $\mathbb{E}_\mu|fg| < \infty$, $\Rightarrow$ $\mathbb{E}_\mu fg \geq \mathbb{E}_\mu f \, \mathbb{E}_\mu g$

proof For any increasing set A, take $X \sim \frac{\mu|_A}{\mu(A)}$ $Y \sim \mu$; then X s.d. Y, and in particular $\mathbb{E} g(X) \geq \mathbb{E} g(Y)$

$(f, g > 0)$ In words, $\mathbb{E}_\mu g \leq \frac{\mathbb{E}_\mu g \cdot \mathbb{1}_A}{\mathbb{P}_\mu A}$

Then take $X \sim \tilde{\mu}$, $\frac{\tilde{\mu}}{\mu} = \frac{g}{\mathbb{E}_\mu g}$, $Y \sim \mu$ $\Rightarrow$ $\tilde{\mu}(A) \geq \mu(A)$; then X s.d. Y, and in particular $\mathbb{E} f(X) \geq \mathbb{E} f(Y)$

In words, $\mathbb{E}_\mu f \leq \mathbb{E}_{\tilde{\mu}} f = \frac{\mathbb{E}_\mu fg}{\mathbb{E}_\mu g}$

FKG: $\Omega = \mathcal{A}^d$, when $\mathcal{A}$ is totally ordered (then Ω poset)

If a measure μ satisfies $\underbrace{\mu(w \vee w')}_{\text{entry-wise max}} \underbrace{\mu(w \wedge w')}_{\text{entry-wise min}} \geq \mu(w)\mu(w')$ (FKG condition)

then μ has pos. asso.

e.g. μ is a product measure. $\mu(w \vee w')\mu(w \wedge w') = \prod_{i=1}^{d} \mu_i(w_i \vee w_i')\mu_i(w_i \wedge w_i') = \prod_{i=1}^{d} \mu_i(w_i)\mu_i(w_i') = \mu(w)\mu(w')$

example: In Erdős-Rényi graph $G(n,p)$

A: $G(n,p)$ is connected

B: chromatic number ≥ 4

$\mathbb{P}[A \cap B] \geq \mathbb{P}[A]\mathbb{P}[B]$

proof of FKG (Assume $\mathcal{A}$ finite, for correctness)

Take $A \subseteq \Omega$ an increasing set; $X \sim \frac{\mu|_A}{\mu(A)}$, $Y \sim \mu$

suffices to show that X s.d. Y.

idea: coupled Markov chain

Glauber dynamics for $\tilde{X} = (\tilde{X}_1, \cdots, \tilde{X}_d)$, $\tilde{Y} = (\tilde{Y}_1, \cdots, \tilde{Y}_d)$

chose $i \in \{1, \cdots, d\}$ uniformly random

replace $\tilde{X}_i$ by $\tilde{X}_i' \sim \frac{\mu|_A}{\mu(A)}(\cdot \mid \tilde{X}_1, \cdots, \tilde{X}_{i-1}, \tilde{X}_{i+1}, \cdots, \tilde{X}_d)$

$\tilde{Y}_i$ by $\tilde{Y}_i' \sim \mu(\cdot \mid \tilde{Y}_1, \cdots, \tilde{Y}_{i-1}, \tilde{Y}_{i+1}, \cdots, \tilde{Y}_d)$

Law of $\tilde{X} \to \frac{\mu|_A}{\mu(A)}$

Law of $\tilde{Y} \to \mu$ since they are the unique stationary measures of these Markov chains

Let $\tilde{X} \geq \tilde{Y}$ initially, can ensure $\tilde{X} \geq \tilde{Y}$ after each step.

Indeed, by FKG condition, for $x_1 \geq y_1, \cdots, x_{i-1} \geq y_{i-1}, x_{i+1} \geq y_{i+1}, \cdots, x_n \geq y_n$

$\mu(\cdot \mid x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$ s.d. $\mu(\cdot \mid y_1, \cdots, y_{i-1}, y_{i+1}, \cdots, y_n)$ (since for any $a \geq b$, $\frac{\mu(x_1, \cdots, x_{i-1}, a, x_{i+1}, \cdots, x_n)}{\mu(x_1, \cdots, x_{i-1}, b, x_{i+1}, \cdots, x_n)} \geq \frac{\mu(y_1, \cdots, y_{i-1}, a, y_{i+1}, \cdots, y_n)}{\mu(y_1, \cdots, y_{i-1}, b, y_{i+1}, \cdots, y_n)}$)

Also $\frac{\mu|_A}{\mu(A)}(\cdot \mid x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$ s.d. $\mu(\cdot \mid x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$

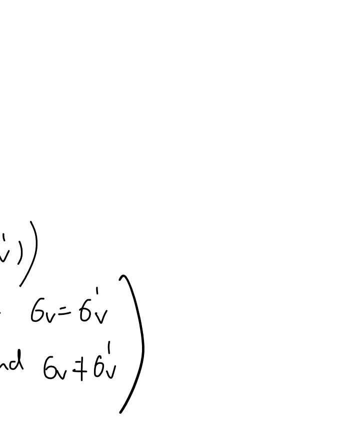
$\Rightarrow \tilde{X} \geq \tilde{Y}$ at time ∞, X s.d. Y.

- Another example: Ising model (ferromagnetic)

In a finite graph G, $\{\sigma_v\}_{v \in V} \in \{\pm 1\}^V$

measure μ on $\{\pm 1\}^V$

s.t. $\mu(\{\sigma_v\}_{v \in V}) = \frac{1}{Z} \cdot \exp\left(\beta \sum_{u \sim v} \sigma_u \sigma_v\right)$



Claim: $\mathbb{E}[\sigma_u \sigma_v] \geq 0$ ($\forall u, v$.)

proof μ satisfies FKG condition

$w = \{\sigma_v\}_{v \in V}$, $w' = \{\sigma_v'\}_{v \in V}$

$\mu(w \vee w')\mu(w \wedge w')$

$= \frac{1}{Z^2} \exp\left(\beta \sum_{u \sim v} (\sigma_u \vee \sigma_u')(\sigma_v \vee \sigma_v') + (\sigma_u \wedge \sigma_u')(\sigma_v \wedge \sigma_v')\right)$

$\begin{pmatrix} = \sigma_u \sigma_v + \sigma_u' \sigma_v', & \text{if } \sigma_u = \sigma_u' \text{ or } \sigma_v = \sigma_v' \\ = 2 \geq \sigma_u \sigma_v + \sigma_u' \sigma_v' & \text{if } \sigma_u \neq \sigma_u' \text{ and } \sigma_v \neq \sigma_v' \end{pmatrix}$

$\geq \mu(w)\mu(w')$

$\Rightarrow \mathbb{E}[\sigma_u \sigma_v] \geq \underbrace{\mathbb{E}[\sigma_u]\mathbb{E}[\sigma_v] = 0}_{\text{by symmetry}}$

- Example: $(X_{ij})_{i,j=1}^{n}$ be $n \times n$ matrix with independent random entries

$\|X\|_2$ operator norm & $\mathrm{tr} X$ positively correlated.

$(\mathbb{P}[\|X\|_2 > a, \mathrm{tr} X > b] \geq \mathbb{P}[\|X\|_2 > a]\mathbb{P}[\mathrm{tr} X > b])$

- First-passage percolation: $T(u,v) = \min_{\gamma} \sum_{e \in \gamma} W_e$. $\{W_e\}_{e \in E}$ independent

$T(u,v)$, $T(u',v')$ positively correlated

$(\mathbb{P}[T(u,v) > a, T(u',v') > b] \geq \mathbb{P}[T(u,v) > a]\,\mathbb{P}[T(u',v') > b])$