Voter model

Saturday, May 10, 2025   2:48 PM

Next 3 weeks: Interacting Particle System (via examples)

①. Voter model / Coalescing random walk

②. Contact process / Infection model

③. Exclusion process (1D)

Problems of interests: stationary distribution? Dynamics? Local/Large scale

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Voter model

(Continuous time) Markov chain, state space $\{0,1\}^V$; V is at most countable

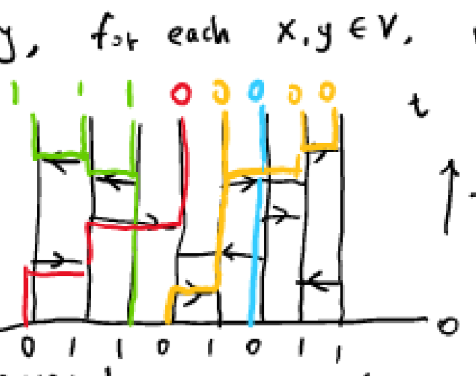
Given a configuration $\eta: V \to \{0,1\}$: then for any $x \in V$, if $\eta(x)=0$, $\to 1$ with rate $\sum_{y\in V, \eta(y)=1} P(x,y)$

if $\eta(x)=1$, $\to 0$ with rate $\sum_{y \in V, \eta(y)=0} P(x,y)$

Here $P(x,y) \geq 0$ are the "activation rates" (assume $\sum_y P(x,y)=1$ for any x)

(Alternatively, for each $x,y \in V$, with rate $P(x,y)$, $\eta(x)$ changes to follow $\eta(y)$)

Duality:



$\uparrow$ time

Follow the opposite direction of arrows.

$\eta_t(x)$: random walk with jump rate P, starting from x at time t; pick the value at time 0.

$\eta_t(x_1), \eta_t(x_2), \cdots, \eta_t(x_k)$: random walks with jump rate P, starting from $x_1, \cdots, x_k$

Walks coalescence upon collision.

V finite & connected: consensus (i.e. all 0 or 1); question: how long it takes?

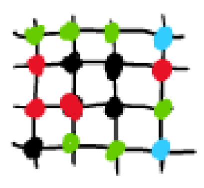
via duality, the same as coalescing time of coalescing random walks (CRW), one from each vertex.

V infinite: may exist "non-trivial" stationary distribution.

$V=\mathbb{Z}^d$, $P(x,y)=\frac{1}{2d}\mathbb{1}_{x\sim y}$; construct stationary as follows: take $0\leq p\leq 1$, and start with $\eta_0 \sim$ i.i.d. Bernoulli(p); take $\lim_{t\to\infty} \eta_t$ (weak limit)

Why this limit exists? Alternative description of η_t, as random clustering.

i.e. consider CRW, starting from one walker at each vertex. Random clustering: x,y in the same cluster iff walkers from x,y coalesce before time t.



For each cluster, sample Bernoulli(p) (independently for all clusters), assign to each x in cluster.

- As $t\uparrow\infty$, clusters combine, converges to time ∞ random clustering (i.e. x,y in the same cluster iff walkers from x,y ever coalesce)

$\Rightarrow$ $d=1,2$, (recurrent) all x in the same cluster; $p\mu_1+(1-p)\mu_0$

$d=3$, (transient) a nontrivial random clustering of $\mathbb{Z}^d$; μ_p a nontrivial stationary measure for voter model.

Are these all the stationary measures?

Of course: any linear combinations of stationary measures is still a stationary measure.

Question: extremal stationary measures? (i.e. those μ s.t. if $\mu=\alpha\mu'+(1-\alpha)\mu''$, necessarily $\mu'=\mu''=\mu$)

<u>Theorem</u> For $d=1,2$, μ_0, μ_1 are all the extremal stationary measures

For $d\geq 3$, $\{\mu_p\}_{p\in[0,1]}$ are all the extremal stationary measures.

①. These measures are extremal

$d=1,2$, obvious

$d\geq 3$, Suppose that $\mu_p=\alpha\mu'+(1-\alpha)\mu''$, want to show $\mu'=\mu''=\mu_p$.

<u>Claim(i)</u> For $\eta_0\sim\mu_p$, $\mathbb{E}[\eta_t(x)|\eta_0] \xrightarrow[t\to\infty]{} p$ in probability.

<u>proof</u>. Since $\mathbb{E}[\mathbb{E}[\eta_t(x)|\eta_0]]=\mathbb{E}[\eta_t(x)]=p$, it suffices to show that $\mathbb{E}[\mathbb{E}[\eta_t(x)|\eta_0]^2]\to p^2$ (as $t\to\infty$)

This $=\sum_{y,z} P_t(x,y)P_t(x,z)\mathbb{P}[\eta_0(y)=\eta_0(z)=1] = p^2 + p(1-p)\underbrace{\sum_{y,z} P_t(x,y)P_t(x,z)\mathbb{P}[y,z \text{ same cluster}]}_{=\mathbb{P}[\text{two independent walkers from } x \text{ meet } \infty \text{ times}]=0} \to p^2$

<u>Claim(ii)</u> For any stationary measure μ, suppose that

$\mathbb{E}[\eta_t(x)|\eta_0\sim\mu]\to p$ in probability, for any $x\in\mathbb{Z}^d$

then necessarily $\mu=\mu_p$.

<u>proof</u> Suffices to show that, for any finite $A\subseteq\mathbb{Z}^d$, $\mathbb{P}_\mu[\eta(A)=1]=\mathbb{P}_{\mu_p}[\eta(A)=1]$.

Consider CRW with one walker starting from each $x\in A$.

$\mathbb{P}_\mu[\eta(A)=1]=\mathbb{E}[\mathbb{P}_\mu[\eta(B_t)=1]]$ B_t = set of remaining walkers at time t

$\approx \mathbb{E}[\prod_{x\in B_t}\mathbb{E}[\eta_s(x)|\eta_0\sim\mu]] + \text{error}(t)$; $\text{error}(t)\to 0$ as $t\to\infty$

$=\mathbb{E}[p^{|B_t|}]+\text{error}(t)$ (send $s\to\infty$)

Same for μ_p ; $\Rightarrow \mathbb{P}_\mu[\eta(A)=1]=\mathbb{P}_{\mu_p}[\eta(A)=1]$

From Claim (i), $\mathbb{E}[\eta_t(x)|\eta_0\sim\mu']\to p$ in prob. ; from Claim (ii), $\mu',\mu''=\mu_p$.

$\mathbb{E}[\eta_t(x)|\eta_0\sim\mu'']\to p$

② All extremal stationary are thus given.

$d=1,2$, for any stationary μ, necessarily $\mathbb{P}_\mu(\eta(x)=\eta(y))=1$ for any $x,y\in\mathbb{Z}^d \Rightarrow \mu=p\mu_1+(1-p)\mu_0$ for some p.

$d\geq 3$. take extremal stationary μ; want to use claim (ii).

By stationarity, $x\mapsto\mathbb{E}_\mu[\eta(x)]$ is harmonic & bounded $\Rightarrow$ constant (denoted by p; assume that $0<p<1$)

Next: show that $\mathbb{E}[\eta_t(x)|\eta_0\sim\mu]\to p$ in prob.

Take any $z\in\mathbb{Z}^d$, let $\mu_{z,0}$ be the measure of μ conditional on $\eta(z)=0$; and

$\mu_{z,1}$ be - - - - - $=1$.

$\Rightarrow \mu=p\mu_{z,1}+(1-p)\mu_{z,0}$

Run Voters starting from $\mu_{z,1}$ & $\mu_{z,0}$ $\Rightarrow$ both converge to μ, by stationarity of μ

(can take any sub-sequential limits μ_1^*, μ_0^*; then both are stationary, and $\mu=p\mu_1^*+(1-p)\mu_0^*$; therefore $\mu_0^*=\mu_1^*=\mu$)

$\Rightarrow \mathbb{P}_\mu(\eta(x)=1)=\lim_{t\to\infty}\sum_y P_t(x,y)\frac{\mathbb{P}_\mu(\eta(y)=\eta(z)=1)}{\mathbb{P}_\mu(\eta(z)=1)} \Rightarrow \lim_{t\to\infty}\sum_y P_t(x,y)\mathbb{P}_\mu(\eta(y)=\eta(z)=1)=p^2 \Rightarrow \lim_{t\to\infty}\sum_{y,z}P_t(x,y)P_t(x,z)\mathbb{P}_\mu(\eta(y)=\eta(z)=1)=p^2$

By Claim (ii), $\mu=\mu_p$

For general Voter model

Harmonic function $h: V\to[0,1]$ ($h(x)=\sum_y P(x,y)h(y)$ for all $x\in V$)

Stationary measure μ_h: start from Bernoulli$(h(x))$, independently for all $x\in V$, run Voter model to time ∞

(limit exists, since $\mathbb{P}[\eta_t(A)=1]$ is non-decreasing in t for any finite $A\subseteq V$)

When is μ_h extremal? If the following holds:

Take any $x,y\in V$, and independent random walks starting from x,y; then almost surely, either $X(t)\neq Y(t)$ for all t large enough or $\lim_{t\to\infty} h(X(t))=0$ or 1 almost surely

Also, μ_h with such h gives all the extremal stationary measures.

(Back to $\mathbb{Z}^d$, $h\equiv p$ are all the harmonic functions; extremal for all p if transient for $p=0,1$ if recurrent)

On the consensus time for finite V. (start with mutually different opinions)

Via duality: same as the time when one walker remains in CRW.

complete graph K_n, each $P(x,y)=1$: $\sum_{k=2}^n \mathrm{Exp}(\binom{k}{2})$ (each $\mathrm{Exp}(\binom{k}{2})$ = time to reduce from k to $k-1$)

(Kingman's coalescence)

In particular, # walkers $\sim\frac{2}{t}$ for t small; $\mathbb{E}T_{coal}=\sum_{k=2}^n\frac{1}{\binom{k}{2}}=2-\frac{2}{n}$

More generally, when $t_{mix}\ll t_{meet}$ $\hookrightarrow \mathbb{E}$[two uniformly started walkers to meet]

$\underset{\text{total-variation}}{T_{coal}\approx} t_{meet}\cdot\sum_{k=2}^n \mathrm{Exp}(\binom{k}{2})$ (Cox, van den Berg-Kesten, Oliveira, Hermon-Li-Yao-Z.)

Walkers at time $t\approx\frac{2t_{meet}}{t}$

For example: $V=(\mathbb{Z}/m\mathbb{Z})^d$ for $d\geq 2$, $t_{mix}=\Theta(m^2)$, $t_{meet}=\Theta(m^2\log m)$ $d=2$; $\Theta(m^d)$ $d\geq 3$

$(\Theta(m^2)$ $d=1)$

V= random d-reg graph with n vertices: $t_{mix}=\Theta(\log(n))$, $t_{meet}=\Theta(n)$