MA 17: HOW TO SOLVE IT
HANDOUT 5: POLYNOMIAL AND ALGEBRAIC METHODS

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Roots and degree. A nonzero polynomial of degree d has at most d distinct roots.

Problem 1. (USAMO 1975, Problem 3)
If $P(x)$ denotes a polynomial of degree n such that $P(k) = k/(k+1)$ for $k = 0, 1, 2, \ldots, n$, determine $P(n+1)$.
Hint: What are the roots of $(x+1)P(x) - x$? What happens at $x = -1$?

Problem 2.
For distinct integers $a_1, \ldots, a_n$, prove that

$$\prod_{j=1}^{n}(x - a_j) - 1$$

has no factorization into two nonconstant polynomials in $\mathbb{Z}[x]$.
Hint: If the factors are f and g, evaluate $f + g$ at each a_j.

Finite differences and coefficients. Define $\Delta F(t) = F(t+1) - F(t)$. If F is a nonconstant polynomial of degree d, then ΔF has degree $d-1$; in particular, $\Delta^{d+1} F = 0$.

Problem 3. (Putnam 2019, B5)
Let F_m be the mth Fibonacci number, defined by $F_1 = F_2 = 1$ and $F_m = F_{m-1} + F_{m-2}$ for all $m \geq 3$. Let $p(x)$ be the polynomial of degree 1008 such that $p(2n+1) = F_{2n+1}$ for $n = 0, 1, 2, \ldots, 1008$. Find integers j and k such that $p(2019) = F_j - F_k$.
Hint: Put $q(t) = p(2t+1)$ and compare finite differences of $q(t)$ and F_{2t+1}.

Problem 4. (Putnam 2022, A2)
Let n be an integer with $n \geq 2$. Over all real polynomials $p(x)$ of degree n, what is the largest possible number of negative coefficients of $p(x)^2$?
Hint: For the upper bound, examine coefficients successively from the constant term. For a construction, try a polynomial with positive end coefficients and equal negative middle coefficients.

Complex numbers. On the unit circle, $\overline{z} = 1/z$. The nth roots of unity are $1, \zeta, \ldots, \zeta^{n-1}$, where $\zeta = e^{2\pi i/n}$, and

$$x^n - 1 = \prod_{j=0}^{n-1}(x - \zeta^j), \qquad \sum_{j=0}^{n-1}\zeta^{jk} = \begin{cases} n, & n \mid k, \\ 0, & n \nmid k. \end{cases}$$

Problem 5.
A regular n-gon ($n \geq 3$) is inscribed in a circle of radius 1. Fix one vertex A. Compute the product of the distances from A to all the other vertices.
Hint: Put A at 1 in the complex plane, factor $(x^n - 1)/(x - 1)$, and substitute $x = 1$.

Problem 6. (Putnam 2018, B2)
Let n be a positive integer, and let $f_n(z) = n + (n-1)z + (n-2)z^2 + \cdots + z^{n-1}$. Prove that f_n has no roots in the closed unit disk $\{z \in \mathbb{C} \colon |z| \leq 1\}$.
Hint: Multiply by $1 - z$. When can equality hold in the triangle inequality?

Linear algebra. There are at most d linearly independent vectors in a space of dimension d.

Problem 7.
Let $A_1, \ldots, A_m$ be subsets of $\{1, \ldots, n\}$. Suppose that each $|A_i|$ is odd, while $|A_i \cap A_j|$ is even whenever $i \neq j$. Prove that $m \leq n$.
Hint: Consider the incidence vectors over $\mathbb{F}_2 = \{0, 1\}$, which give an orthogonal matrix Q, i.e., $Q^T Q = I$.

Problem 8. (Putnam 2019, B3)
Let Q be an n-by-n real orthogonal matrix, and let $u \in \mathbb{R}^n$ be a unit column vector (that is, $u^T u = 1$). Let $P = I - 2uu^T$, where I is the n-by-n identity matrix. Show that if 1 is not an eigenvalue of Q, then 1 is an eigenvalue of PQ.
Hint: What must the determinant of an orthogonal matrix be if 1 is not an eigenvalue?

The Cayley–Hamilton theorem. If $\chi_A(t) = \det(tI - A)$ is the characteristic polynomial of a square matrix A, then $\chi_A(A) = 0$.

Problem 9.
Let A be a 2×2 complex matrix, and let $u, v \in \mathbb{C}$. Suppose that $uI + vA$ is invertible. Prove that there are $u', v' \in \mathbb{C}$ such that

$$(uI + vA)^{-1} = u'I + v'A.$$

Hint: Apply Cayley–Hamilton to $B = uI + vA$, then multiply by B^{-1}.

The Perron–Frobenius theorem (positive matrices). A real square matrix whose entries are all strictly positive has a positive eigenvalue ρ and an eigenvector whose coordinates are all strictly positive. The eigenvalue ρ is simple and satisfies $|\lambda| < \rho$ for every other eigenvalue λ.

Problem 10. (IMO 2003 Shortlist, A1)
Let a_{ij}, $i = 1, 2, 3$; $j = 1, 2, 3$ be real numbers such that a_{ij} is positive for $i = j$ and negative for $i \neq j$. Prove that there exist positive real numbers c_1, c_2, c_3 such that the numbers

$$a_{11}c_1 + a_{12}c_2 + a_{13}c_3, \qquad a_{21}c_1 + a_{22}c_2 + a_{23}c_3, \qquad a_{31}c_1 + a_{32}c_2 + a_{33}c_3$$

are all negative, all positive, or all zero.
Hint: Choose s so that every entry of $sI - A$ is strictly positive.

Reminder PSet 5 to be released tomorrow, and due by the end of Nov 18 (on Canvas).