MA 17: HOW TO SOLVE IT
PROBLEM SET 5

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Due: 11:59PM of November 20. Late submissions will not be accepted.

Instructions:

- Please submit solutions to at most 3 problems, written carefully. You are encouraged to work on more, but the TA will provide feedback on only 3. I recommend selecting the 3 for which you most want feedback.
- Collaboration on homework before you have solved at least 3 problems independently is discouraged, though not strictly forbidden. You should make a serious effort to work through each problem independently before turning to classmates, using the internet, or asking AI.
- As long as your submission clearly shows attempts on at least 3 problems, you will earn 8 points toward the final grade. The correctness of your solutions will not affect your points, although feedback will be provided.
- Submission should be made through Canvas.

Problem 1. (Problem 9 in Handout 5)
Let A be a 2×2 complex matrix, and let $u, v \in \mathbb{C}$. Suppose that $uI + vA$ is invertible. Prove that there are $u', v' \in \mathbb{C}$ such that

$$(uI + vA)^{-1} = u'I + v'A.$$

Problem 2.
Let A and B be 3×3 matrices with complex entries. Prove that

$$\det(AB - BA) = \frac{\mathrm{tr}((AB - BA)^3)}{3},$$

where tr denotes the trace of a matrix.

Problem 3. (Putnam 2003, B1)
Do there exist polynomials $a(x), b(x), c(y), d(y)$ such that

$$1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)$$

holds identically?

Problem 4.
Let n be a positive integer and $P(x)$ an nth-degree polynomial with complex coefficients such that $P(0), P(1), \ldots, P(n)$ are all integers. Prove that the polynomial $n!P(x)$ has integer coefficients.

Problem 5. (All-Union Olympiad 1977, Grade 8, Problem 5)
Seven dwarfs are sitting around a circular table. There is a cup in front of each. There is milk in some cups, altogether 3 liters. One of the dwarfs shares his milk uniformly with the other cups. Proceeding counter-clockwise, each of the other dwarfs, in turn, does the same. After the seventh dwarf has shared his milk, the initial content of each cup is restored. Find the initial amount of milk in each cup.

Problem 6.
Prove that for every prime number p, the polynomial

$$P(x) = x^{p-1} + x^{p-2} + \cdots + x + 1$$

is irreducible over $\mathbb{Z}[x]$, meaning that it cannot be written as the product of two nonconstant polynomials with integer coefficients.

Problem 7. (Yugoslav Federal Competition 1980, Year 3, Problem 4)
Find all polynomials $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ with $a_j \in \{-1, 1\}$ for each j, whose all roots are real.

Problem 8.
Consider the $n \times n$ matrix $A = (a_{ij})$ with $a_{ij} = 1$ if $j - i \equiv 1 \pmod{n}$ and $a_{ij} = 0$ otherwise. For real numbers a and b, find the eigenvalues of $aA + bA^T$.

Problem 9. (Putnam 2019, A5)
Let p be an odd prime number, and let $\mathbb{F}_p$ denote the field of integers modulo p. Let $\mathbb{F}_p[x]$ be the ring of polynomials over $\mathbb{F}_p$, and let $q(x) \in \mathbb{F}_p[x]$ be given by

$$q(x) = \sum_{k=1}^{p-1} a_k x^k,$$

where

$$a_k = k^{(p-1)/2} \mod p.$$

Find the greatest nonnegative integer n such that $(x-1)^n$ divides $q(x)$ in $\mathbb{F}_p[x]$.

Problem 10. (Putnam 2018, A2)
Let $S_1, S_2, \ldots, S_{2^n-1}$ be the nonempty subsets of $\{1, 2, \ldots, n\}$ in some order, and let M be the $(2^n - 1) \times (2^n - 1)$ matrix whose (i, j) entry is

$$m_{ij} = \begin{cases} 0 & \text{if } S_i \cap S_j = \emptyset; \\ 1 & \text{otherwise.} \end{cases}$$

Calculate the determinant of M.

Reminder Next Class: Nov 17.