# Constructing extremal stationary distributions for the Voter Model in $d \geq 3$ as factors of IID

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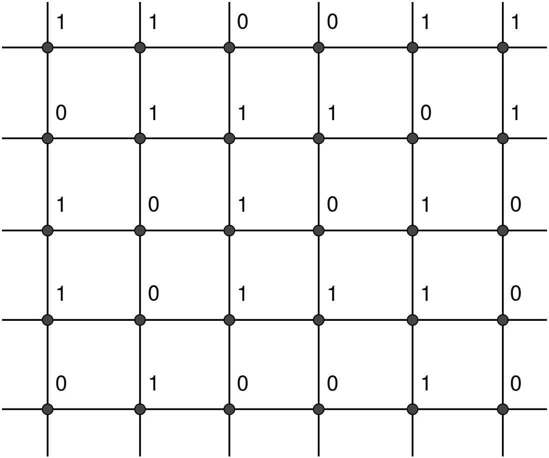
# Schedule

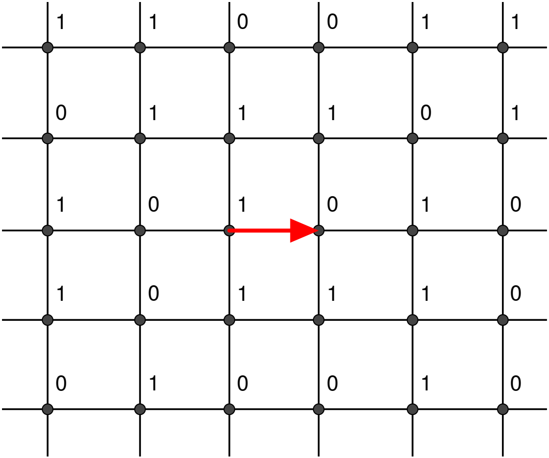
1. Problem & background
2. Proof by construction
3. Further questions

# Problem & background




Voter Model on $\mathbb{Z}^d$: $\eta_x^{(t)}$ flips with rate $\#\{y \sim x : \eta_x^{(t)} \neq \eta_y^{(t)}\}$.

# Voter Model on Lattice

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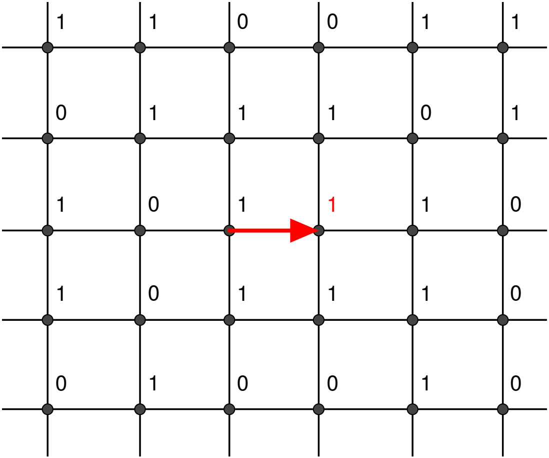


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# Extremal stationary distributions

- This defines a Markov process on $\{0,1\}^{\mathbb{Z}^d}$, with transition operator $\{\mathcal{M}_t\}_{t\in\mathbb{R}_+}$.
- Take initial distribution to be $\rho_p := \mathrm{Bern}(p)^{\mathbb{Z}^d}$, for $p \in [0,1]$.
- Weak limit $\mu_p := \lim_{t\to\infty} \mathcal{M}_t \rho_p$ exists and is a stationary distribution.

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When $d \geq 3$, $\{\mu_p\}_{p\in[0,1]}$ are all the extremal stationary distributions.

Construct $\mathcal{M}_t \rho_p$ via duality:

1. Run colaescing random walk (CRW) $\{A_t\}_{t \in \mathbb{R}_+}$.
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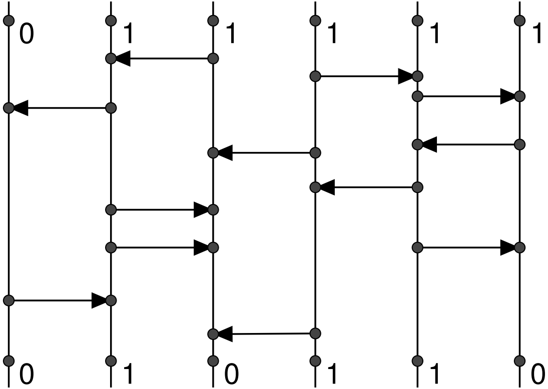

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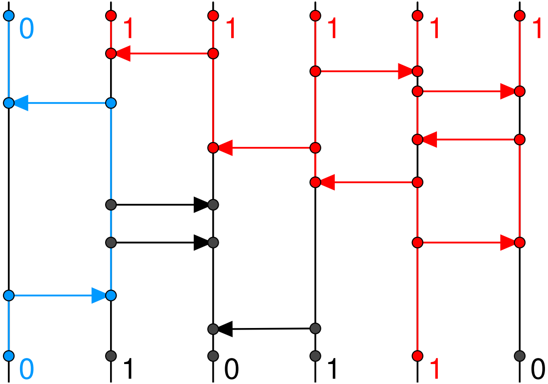

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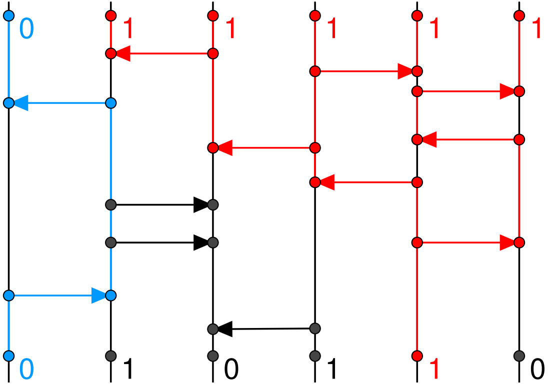


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  so $\mu_p = p\mu_1 + (1-p)\mu_0$.

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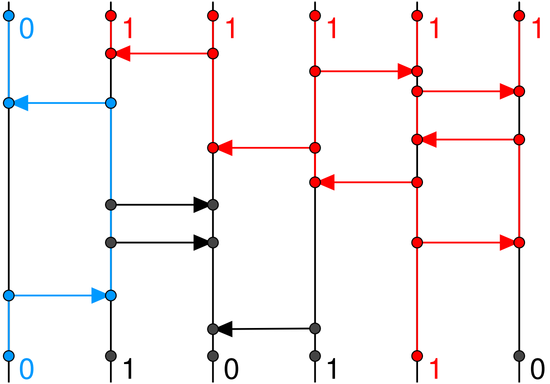


- $d = 1, 2$: random walk is recurrent, there is a unique cluster in CRW as $t \to \infty$, so $\mu_p = p\mu_1 + (1-p)\mu_0$.
- $d \geq 3$: random walk is transient, $\{\mu_p\}_{p\in[0,1]}$ are all the extremal stationary distributions

# Generalized Divide and Color Model

The voter model distributions are in a more general family, studied by [Steif and Tykesson, 2017].

1. Finite or countable set $V$, and a random partition/ equivalence relation (RER) on it.
2. A parameter $p \in [0, 1]$, and color each partition element by 0 or 1, independent $\sim \mathrm{Bern}(p)$.

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1. Voter model $\mathcal{M}_t\rho_p$ and stationary distribution $\mu_p$: the RER is given by CRW.
2. Ising model (and Potts model): the RER is given by the FK percolation, or random cluster model (RCM) via Edwards-Sokal coupling.

# Factor of IID

- IID process over a group: $(Y^G, \nu^G, G)$.
- Factor $\mathcal{F} : (Y^G, \nu^G, G) \to (X, \mu, G)$.
- In our case, $G = \mathbb{Z}^d$ translations, $X = \{0, 1\}^{\mathbb{Z}^d}$.

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Equivalently (in our case), $\eta \sim (\{0, 1\}^{\mathbb{Z}^d}, \mu)$ is a factor of IID if: there is a function $f : Y^{\mathbb{Z}^d} \to \{0, 1\}$, and an IID process $\{\gamma_x\}_{x \in \mathbb{Z}^d}$, such that $\forall x \in \mathbb{Z}^d$, $\eta_x = f(T_x \gamma)$.
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An example that is not a factor of IID: $\eta \equiv 0$ or 1, each with probability $1/2$.

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## An ergodic theory point of view:

- When $Y$ is finite, $(Y^{\mathbb{Z}}, \nu^{\mathbb{Z}})$ is a Bernoulli shift.
- Being a factor of IID $\iff$ (isomorphic to) Bernoulli shift

(by Ornstein theory [Ornstein, 1970a][Ornstein, 1970b] and their generalizations to amenable groups [Ornstein and Weiss, 1987])

If RER is Bernoulli, and each cluster is finite, then the color process is also Bernoulli.
e.g. CRW is Bernoulli $\Longrightarrow$ $\mathcal{M}_t\rho_p$ is Bernoulli (for $t \in (0, \infty)$).

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**Question ([Steif and Tykesson, 2017, Question 7.20 ])**

*Give an example of a divide and color model, which has infinite clusters but the color process is Bernoulli.*

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Steif and Tykesson suggest the stationary distribution of the Voter Model.

**Question ([Steif and Tykesson, 2017, Question 7.18 ])**

*When $d \geq 3$, are the Voter Model extremal stationary distributions Bernoulli shifts?*

We give an affirmative answer to both questions.

**Theorem ([Sly and Z., 2019])**

*When $d \geq 3$, for each $0 \leq p \leq 1$, $\mu_p$ is a factor of IID.*

# Proof by construction

Explicit construct $\mu_p$:

- Take $\eta^{(t_k)} \sim \mathcal{M}_{t_k}\rho_p$ for a growing sequence of time $\{t_k\}_{k\in\mathbb{Z}_+}$
- Each $\eta^{(t_k)}$ is a factor of IID.
- Couple them in a translation invariant way, so that each $\mathbb{P}[\eta_x^{(t_k)} \neq \eta_x^{(t_{k+1})}]$ is small.

Then $\eta^{(t_k)} \xrightarrow{\text{a.s.}} \eta \sim \mu_p$, and $\mu_p$ is a factor of IID.

# General idea

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Couple $\mathcal{M}_t\rho_p$ and $\mathcal{M}_{t+\Delta t}\rho_p$
( $\iff$ Run CRW from $A_t$ to $A_{t+\Delta t}$, plus coloring).

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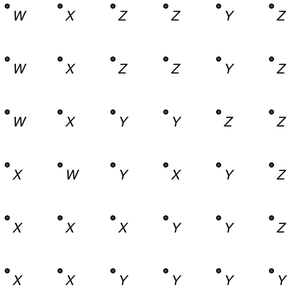
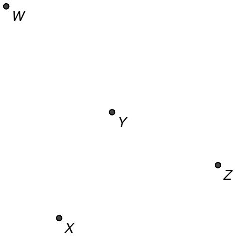



RER of $\mathcal{M}_t \rho_p$




$A_t$

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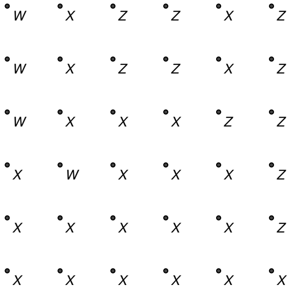
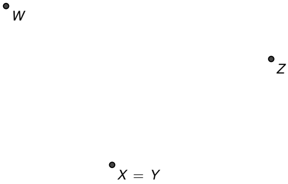



RER of $\mathcal{M}_{t+\Delta t} \rho_p$

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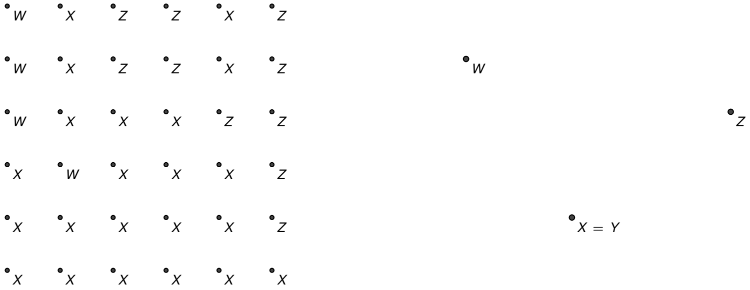


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**CRW + independent coloring does not work:**
**color change infinitely many times.**

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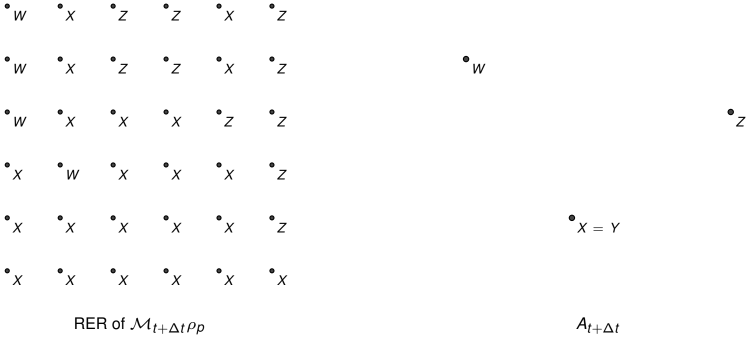


**CRW + independent coloring does not work:**
**color change infinitely many times.**
Need biased coupling based on coloring.

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1. Build a random tree structure: size $\sim 4^k$ at level $k$.
   (need randomness to make it translation invariant)
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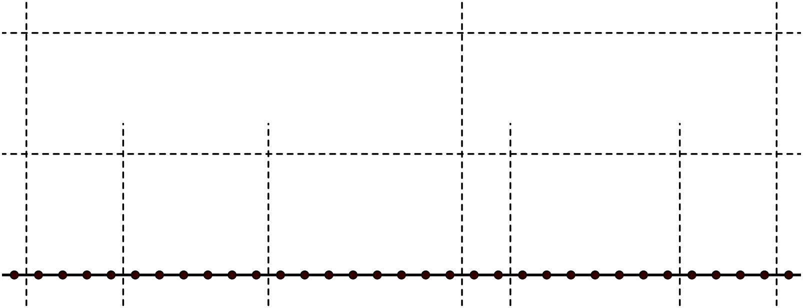

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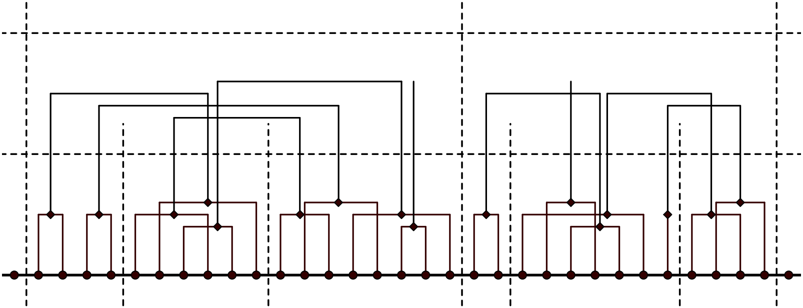

1. Random color by $\text{Bern}\, p$.
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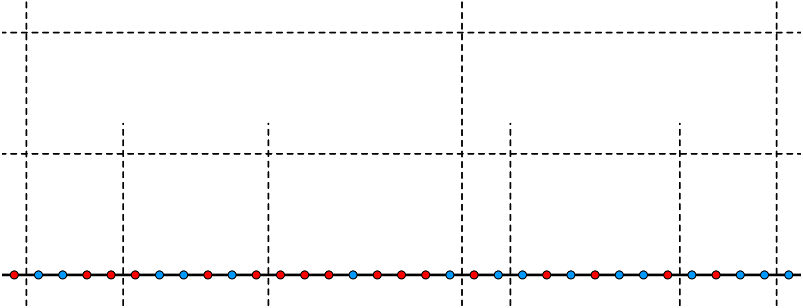
4. 



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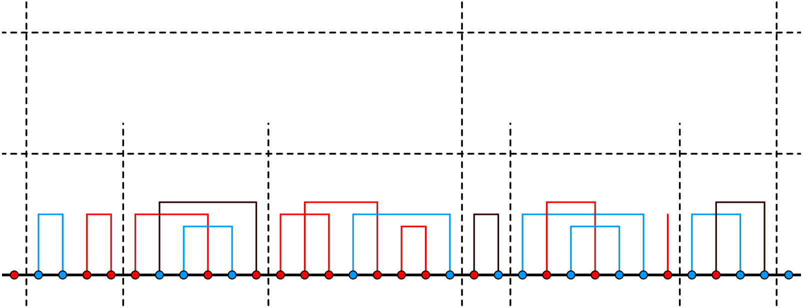

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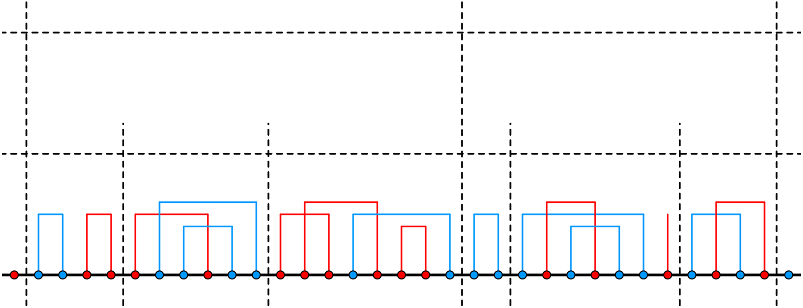

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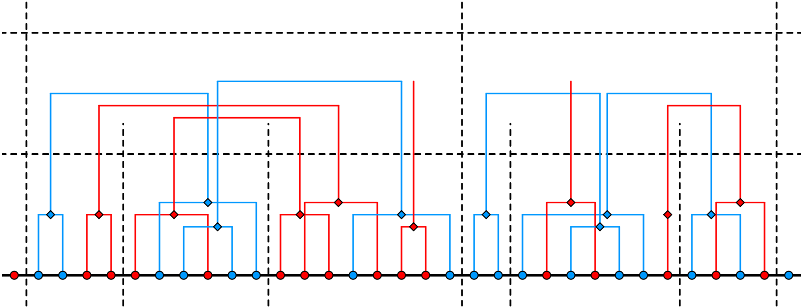

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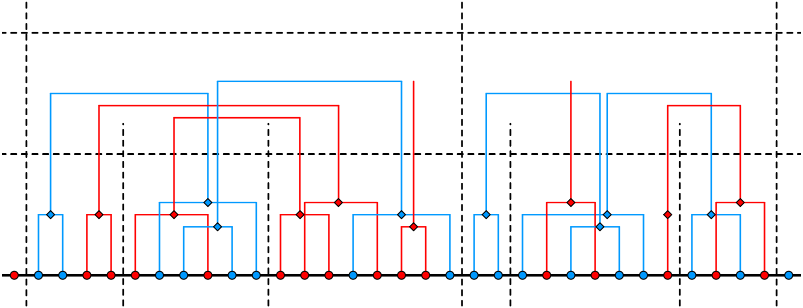

Almost surely, each vertex changes color finitely many times.

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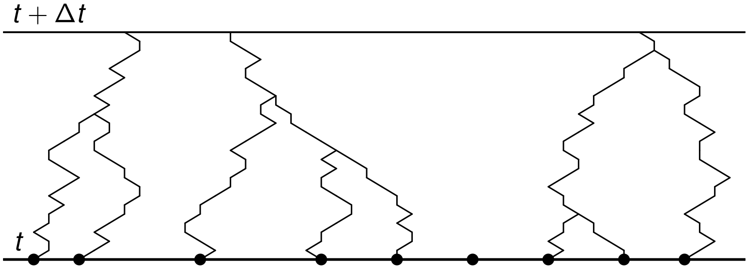
Construct CRW: add paths sequentially.



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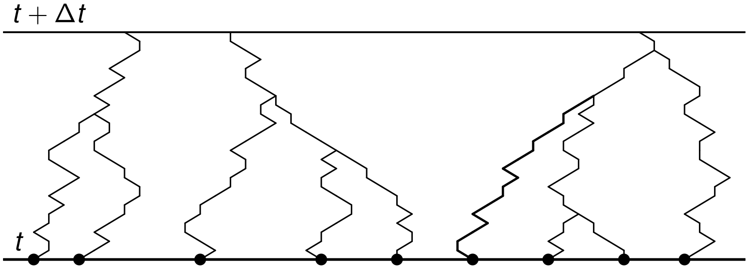

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1. Conditioned on some clusters and coloring.
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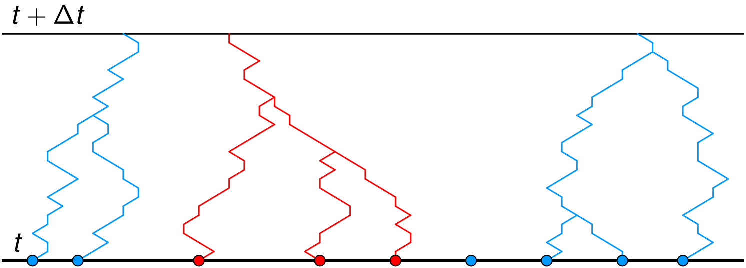

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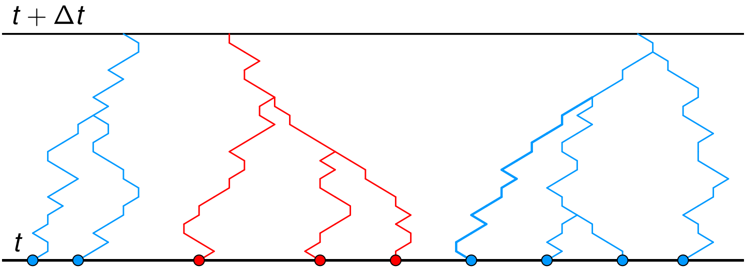

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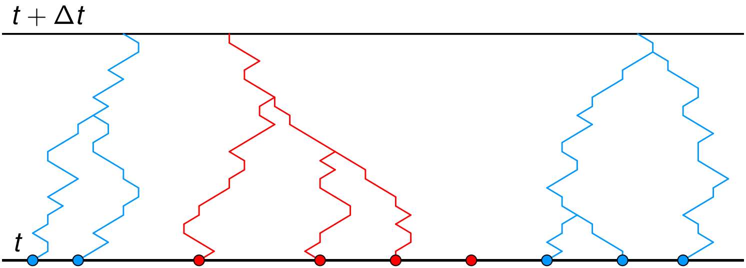

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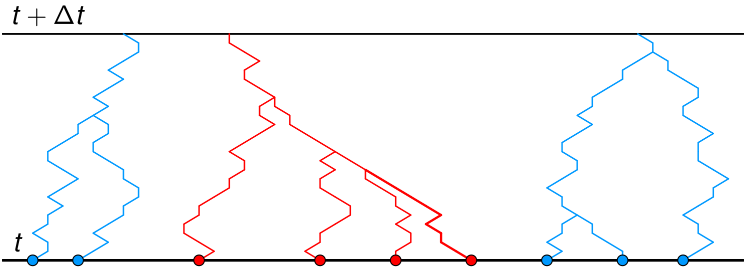

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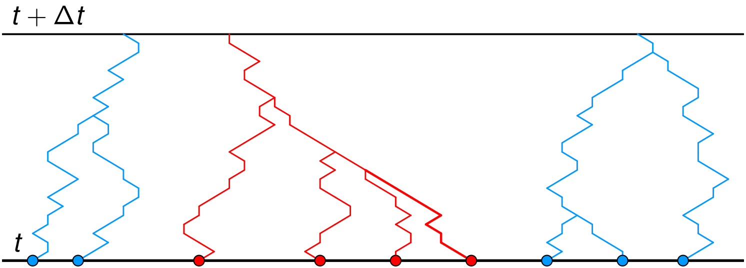


Ideally, we have:

1. Law of colors of walkers at $t$: $\mathrm{Bern}(p)$ independently.
2. Law of the paths: CRW.
3. Law of colors of each cluster between $t$ and $t+\Delta t$: $\mathrm{Bern}(p)$ independently.

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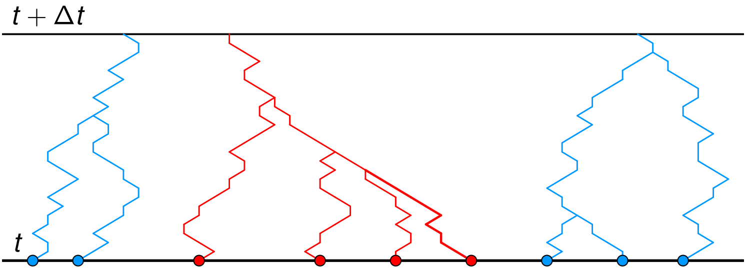


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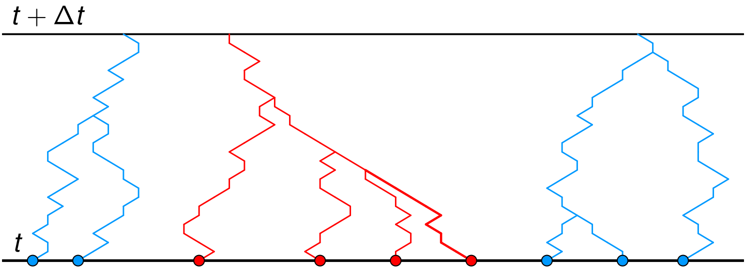


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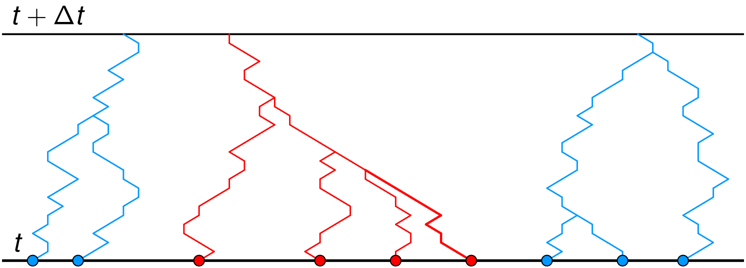
**These cannot hold at the same time unless $\mathbb{P}$[join red cluster] $= p$.**

On average, $\mathbb{P}$[join red cluster] is $p$.
Change color when necessary.

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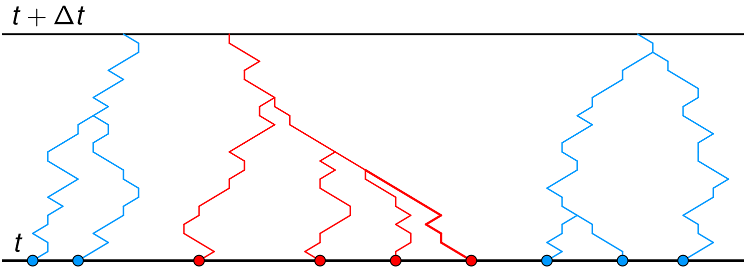


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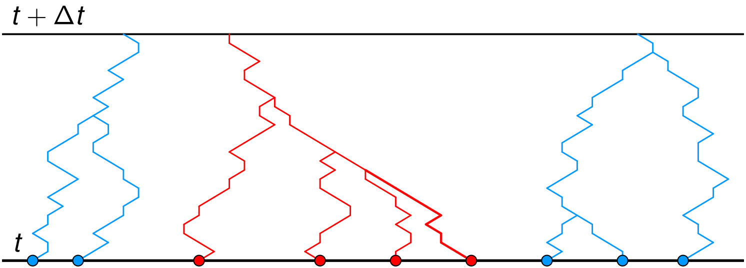


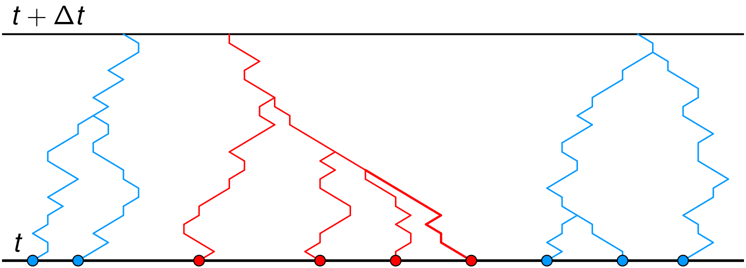

Control $\mathbb{P}$[walker $x$ changes color]:

$\leq \mathbb{E}\left[|\mathbb{P}[x \text{ joins red cluster}|\text{existing colored clusters}] - p|\right]$.

$\leq \mathbb{E}\left[\sum_{\text{cluster}\, Y} \mathbb{P}[x \text{ joins } Y|\text{existing clusters}]^2\right]^{1/2}$

(Cauchy-Schwarz, integrating coloring)

# Couple $\mathcal{M}_t \rho_p$ and $\mathcal{M}_{t+\Delta t}\rho_p$: CRW from $t$ to $t+\Delta t$




Control $\mathbb{P}$[walker $x$ changes color]:

$\leq \mathbb{E}\left[ \left| \mathbb{P}[x \text{ joins red cluster} | \text{existing colored clusters}] - p \right| \right].$

$\leq \mathbb{E}\left[ \sum_{\text{cluster} Y} \mathbb{P}[x \text{ joins } Y | \text{existing clusters}]^2 \right]^{1/2}$

(Cauchy-Schwarz, integrating coloring)

$\leq \mathbb{E}\left[ \sum_{x_1, x_2 \in A_t} \mathbb{P}[x_1, x_2, x \text{ in same cluster in CRW}] \right]^{1/2}$
$= O((t^{-2}(\Delta t)^{3-d/2})^{1/2})$ (expand the clusters into particles)

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## Cannot impose ordering: divide into groups

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Divide into groups:

1. Randomly divide walkers into groups $G_1, G_2, \cdots, G_M$.
2. Each group is sparse: avg. distance $\gg \sqrt{\Delta t}$.
3. Construct colored CRW for these groups sequentially; unlikely for a walker to hit another walker from the same group. (Prob arbitrarily small by taking $M$ large)

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# Exponentially growing times

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$\implies$ under the coupling $\eta^{(2^k)}$ converges almost surely, so $\mu_p$ is a factor of IID.

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- On $\mathbb{Z}^d$, the RER (FK percolation/RCM) has
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  one unique infinite cluster when $\beta > \beta_c$
  $\implies$ Ising model is not a factor of IID.
- On infinite $d$-reg tree,
  the RER has no infinite cluster when $(d-1)\tanh(\beta) \leq 1$
  $\implies$ Ising model is a factor of IID.
  reconstruction is possible when $(d-1)\tanh^2(\beta) > 1$
  $\implies$ Ising model is not a factor of IID.
  intermediate $\beta$: open; conjectured to be factor of IID [Lyons, 2017].

Thank you!

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