Cutoff profile of the colored ASEP

arXiv:2208.13383

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Berkeley Probability Seminar **Oct 12, 2022**

Agenda

- The model of (colored) ASEP
- Background: known results on mixing/cutoff
- Intuition: why GOE Tracy-Widom cutoff profile
- The strategy: finishing times via coupling, symmetry via Hecke algebra
- Further questions

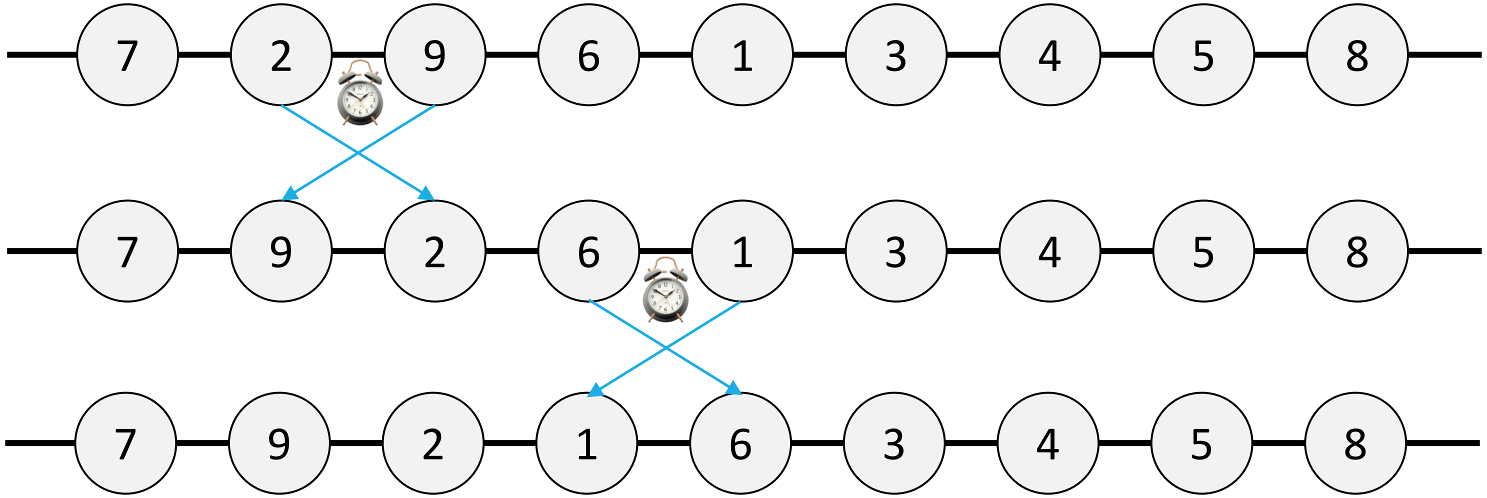
The model

Asymmetric Simple Exclusion Process (ASEP) with colors:

--- a (continuous time) Markov chain on S_N

Rate 1 for $left < right$

Rate q for $left > right$
$(0 \leq q < 1)$



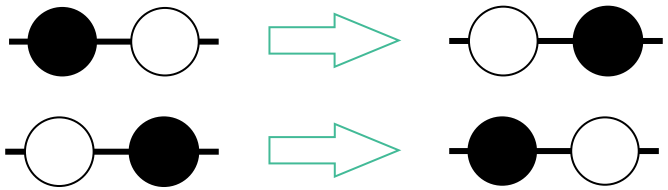
Also called *biased card shuffling* or *random Metropolis scan*.

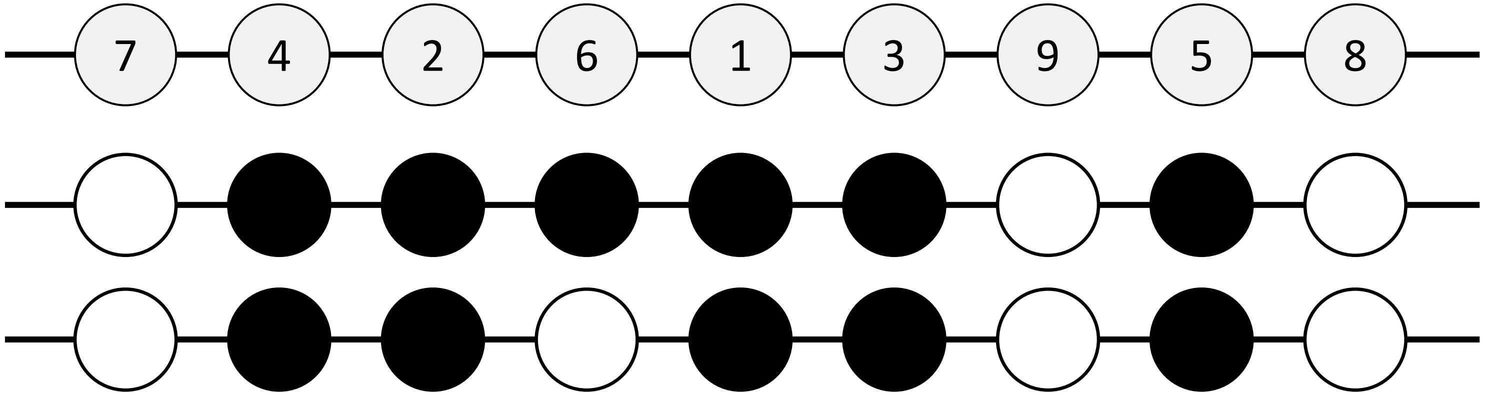
Projections to **ordinary ASEP**

Ordinary ASEP (without colors): particles and holes, segment or $\mathbb{Z}$

Rate 1

Rate q





Projection for colors ≤ 6

Projection for colors ≤ 5

Colored ASEP can be recovered from *all projections*.

Stationary: Mallows measure

For any $\omega \in S_N$,

$$\mathrm{M}_N(\omega) = q^{\kappa(\omega)} \prod_{i=1}^{N} \frac{1-q}{1-q^i}$$

where $\kappa(\omega) = |\{1 \le i < j \le N \colon \omega(i) < \omega(j)\}|$

Ground state: $\omega(i) = N + 1 - i$

Ordinary ASEP: for any $\omega \in \{0,1\}^N$ with $\sum_{i=1}^{N} \omega(i) = M$, (i.e., M particles)

$$\mathrm{M}_N^M(\omega) = q^{\kappa(\omega)} \frac{\prod_{i=1}^{M}(1-q^i) \prod_{i=1}^{N-M}(1-q^i)}{\prod_{i=1}^{N}(1-q^i)}$$

where $\kappa(\omega) = |\{1 \le i < j \le N \colon \omega(i) = 1,\ \omega(j) = 0\}|$

Some previous works

- Diaconis-Ram, 2000 (random Metropolis scan)
- Benjamini-Berger-Hoffman-Mossel, 2002

Mixing time is order N

- Labbé-Lacoin, 2016 cutoff for colored/ordinary ASEP (via hydrodynamics)

speculated $N^{1/3}$ cutoff window, and KPZ-related cutoff profile

Total Variation dist at $(2-\epsilon)N/(1-q) \to 1$
Total Variation dist at $(2+\epsilon)N/(1-q) \to 0$

- Bufetov-Nejjar, 2020

ordinary ASEP: $N^{1/3}$ cutoff window and GUE Tracy-Widom profile

Conjecture: colored ASEP also has $N^{1/3}$ cutoff window, with GOE Tracy-Widom profile

Why $N^{1/3}$ and GOE Tracy-Widom?

Special case: Totally Asymmetric Simple Exclusion Process (TASEP)

$(q = 0)$

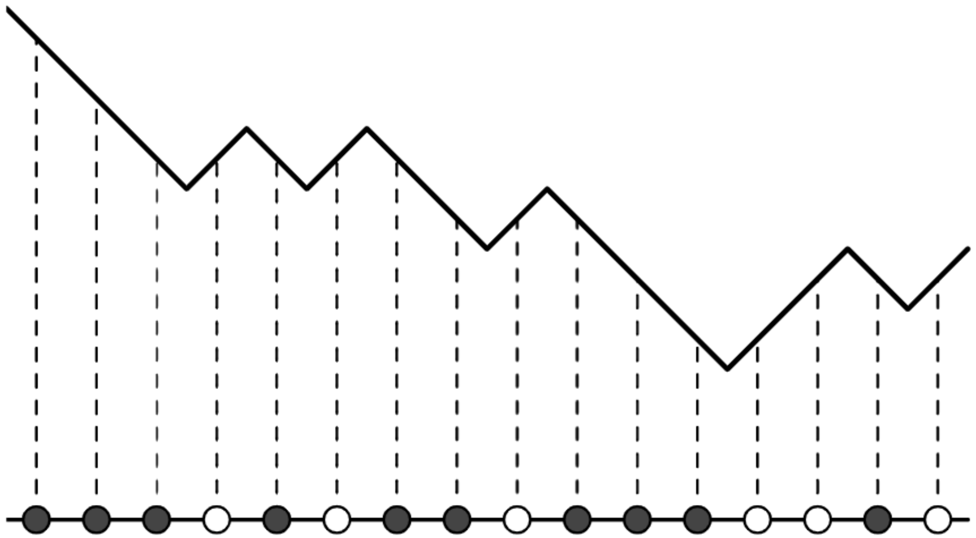
Mallows measure degenerates into an absorbing state:

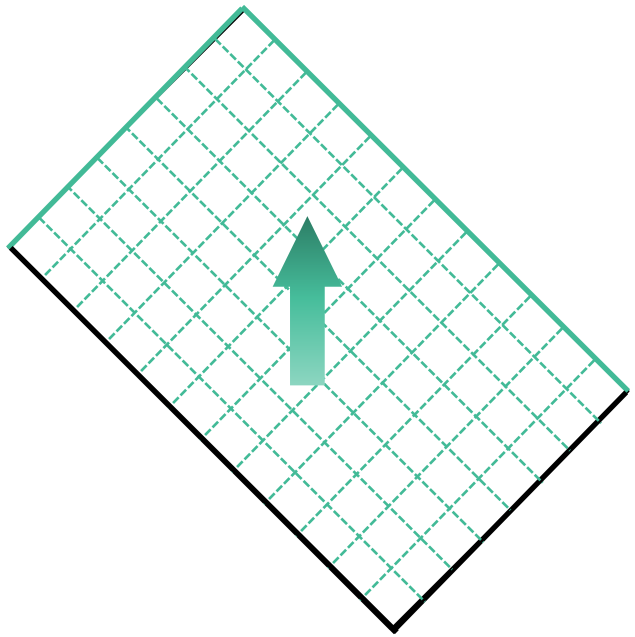
$$\omega(i) = N + 1 - i$$

Total variation mixing ➡ absorbing time

Why $N^{1/3}$ and GOE Tracy-Widom?

Ordinary TASEP: corresponds to a growing surface



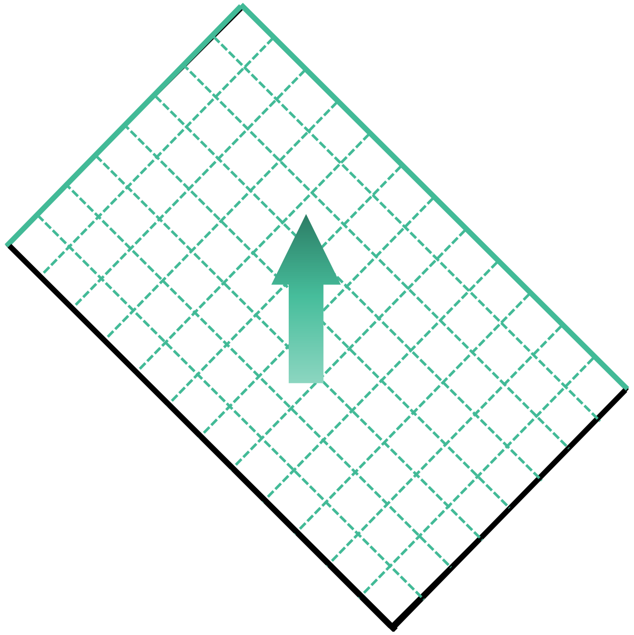


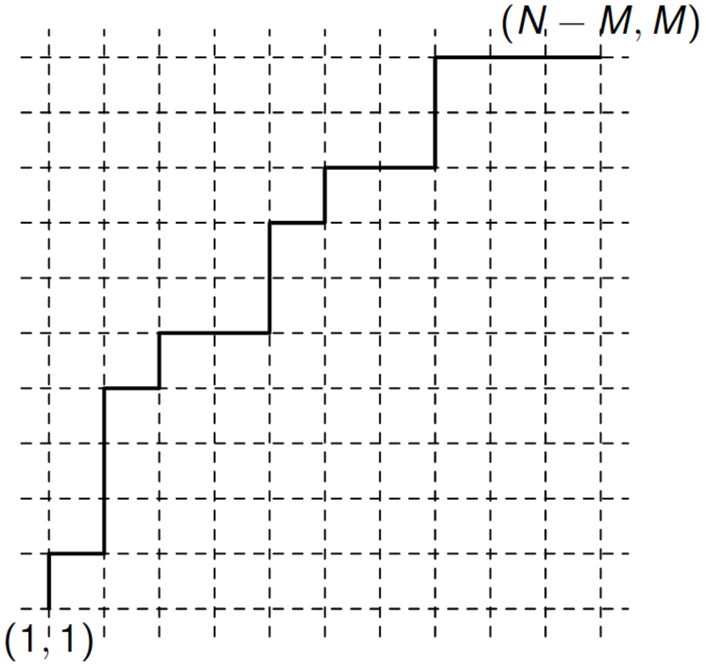
Initially: M particles at the left, $N - M$ holes at the right

Absorbing state: $N - M$ holes at the left, M particles at the right

Why $N^{1/3}$ and GOE Tracy-Widom?

Ordinary TASEP: corresponds to a growing surface





Last Passage Percolation with i.i.d. $\mathrm{Exp}(1)$ weights.

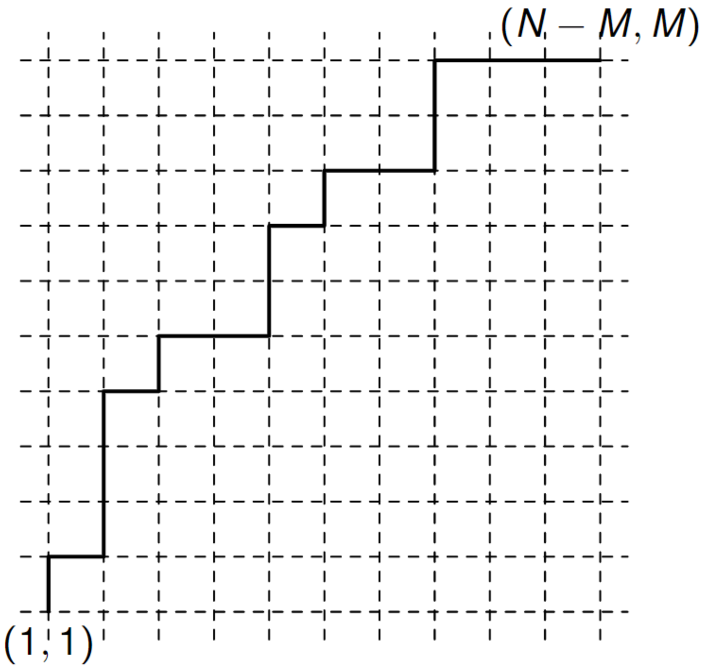
Why $N^{1/3}$ and GOE Tracy-Widom?

Known (Johansson, 1999, via RSK)

if $\frac{M}{N} \to y \in (0,1)$,

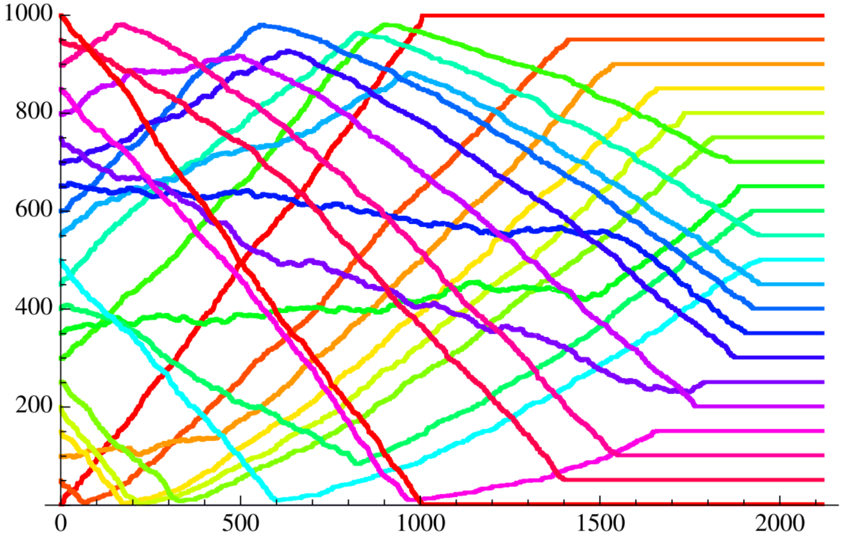
$$\frac{L_{N-M,M} - N\left(1+2\sqrt{y(1-y)}\right)}{N^{1/3}\left(1+2\sqrt{y(1-y)}\right)^{2/3}\left(y(1-y)\right)^{-1/6}} \to \text{GUE Tracy-Widom}$$

Ordinary TASEP absorbing time is linear in N, plus $N^{1/3}$ times GUE Tracy-Widom



Why $N^{1/3}$ and GOE Tracy-Widom?

Colored TASEP (Oriented Swap Process, Angel-Holroyd-Romik, 2008)



Using projections:
finishing time $F_{N,M}$ for each particle has $N^{1/3}$ Tracy-Widom GUE

Question: absorbing time? $A_N = \max_{1 \leq M \leq N} F_{N,M}$

Bufetov-Gorin-Romik, 2020

$\frac{A_N - 2N}{2^{1/3} N^{1/3}} \to$ GOE Tracy-Widom

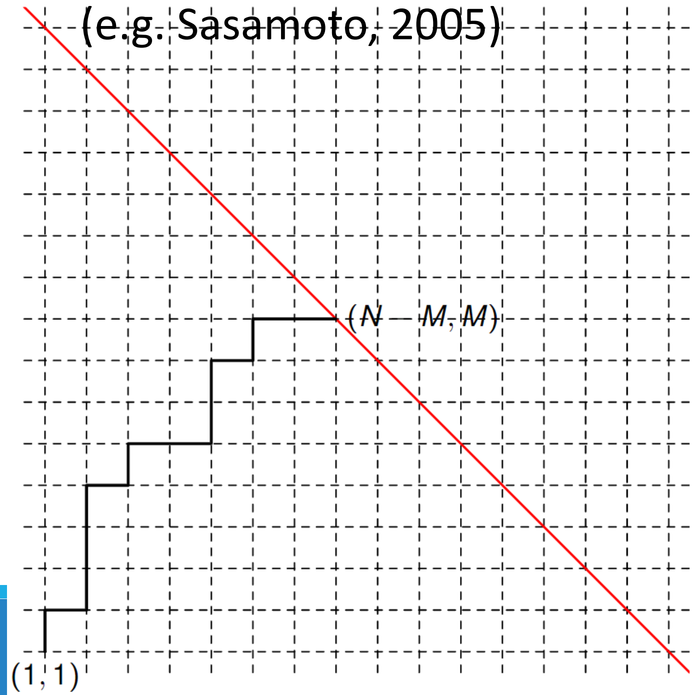
Why $N^{1/3}$ and GOE Tracy-Widom?

Colored TASEP (Oriented Swap Process, Angel-Holroyd-Romik, 2008)

Bufetov-Gorin-Romik, 2020 $\frac{A_N - 2N}{2^{1/3} N^{1/3}} \to$ GOE Tracy-Widom

Note: $A_N = \max_{1 \le M \le N} F_{N,M}$, and each $F_{N,M} = L_{N-M,M}$ in distribution.

Known to be GOE Tracy-Widom (e.g. Sasamoto, 2005)



Conjecture:

$$\{F_{N,M}\}_{M=1}^{N} = \{L_{N-M,M}\}_{M=1}^{N}, \text{ so } A_N = \max_{1 \le M \le N} L_{N-M,M}$$

(Z., 2021) (Bufetov-Gorin-Romik, 2020)

Follow non-trivial shift-invariance (Borodin-Gorin-Wheeler, 2019, Galashin, 2020)

Why $N^{1/3}$ and GOE Tracy-Widom?

	TASEP absorbing time distribution	ASEP cutoff profile
Ordinary (colorless)	GUE Tracy-Widom (Angel-Holroyd-Romik, 2008, using LPP results from Johansson, 1999)	GUE Tracy-Widom (Bufetov-Nejjar, 2020: Hecke algebra, coupling, step initial ASEP results by Tracy-Widom, 2008)
Colored	GOE Tracy-Widom (shift invariance in Bufetov-Gorin-Romik, 2020, using LPP results from Sasamoto, 2005)	Natural guess: GOE Tracy-Widom (Z. 2022)

Theorem. (Z. 2022) Let $W_N^{\lambda}(t)$ denote the law of the S_N colored ASEP with initial configuration λ at time t, then

$$\max_{\lambda \in S_N} \left\| W_N^{\lambda}\left(2(1-q)^{-1}(N+\tau N^{1/3})\right) - M_N \right\|_{TV} \to 1 - F_{GOE}(2^{2/3}\tau)$$

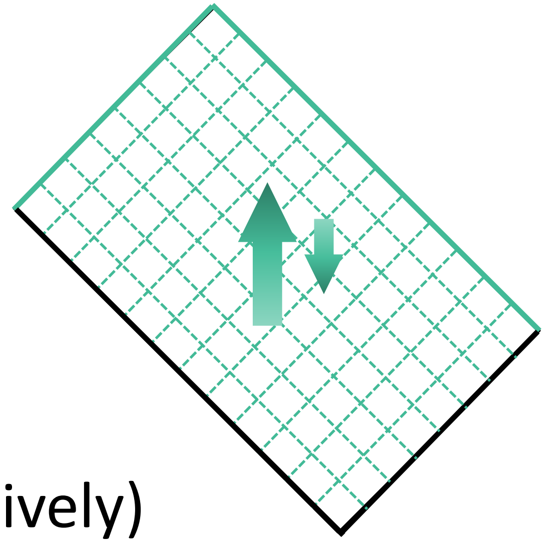
The strategy

Observation: still suffices to consider the ‘finishing times’:

Let $F^*_{N,M}$ be the first time when all numbers $\leq M$ are at locations $> N - M$.

Then total-variation mixing happens roughly at $\max_{1\leq M\leq N} F^*_{N,M}$.

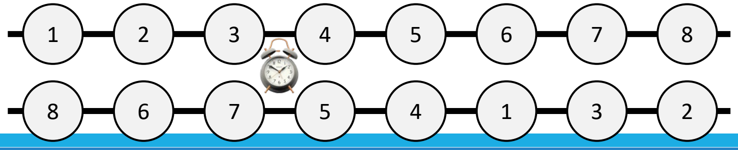
Why?



i. Given $t < \max_{1\leq M\leq N} F^*_{N,M}$, the configuration at time t is ‘unlike’ the (stationary) Mallows measure; so

$$\left\| W_N^{Id}(t) - M_N \right\|_{TV} \geq \mathbb{P}\left[t < \max_{1\leq M\leq N} F^*_{N,M} \right]$$

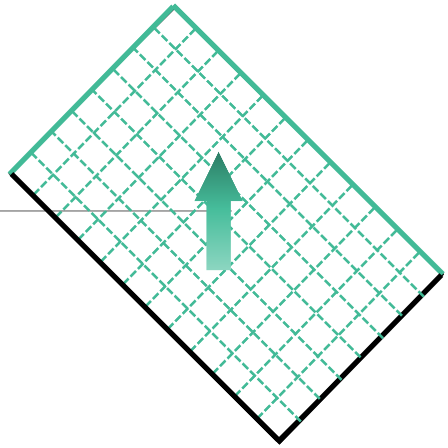
ii. Take the basic coupling (synchronize two-types of alarms respectively) with the stationary process



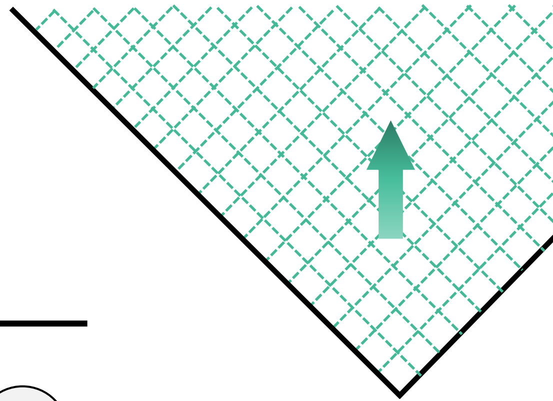
Since time $F^*_{N,M}$, the $\leq M$ projections are the same.

Since time $\max_{1\leq M\leq N} F^*_{N,M}$, all the same.

The strategy: $\max_{1 \le M \le N} F^*_{N,M}$

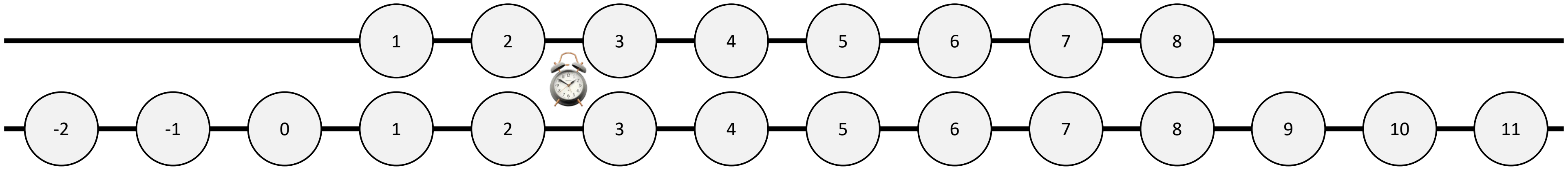


1. Truncation: a segment versus $\mathbb{Z}$
2. Shift-invariance Existing from Borodin-Gorin-Wheeler, 2020 or Galashin, 2021
3. Use known LPP/TASEP or ASEP asymptotic results
   Existing (Quastel-Sarkar, 2020)



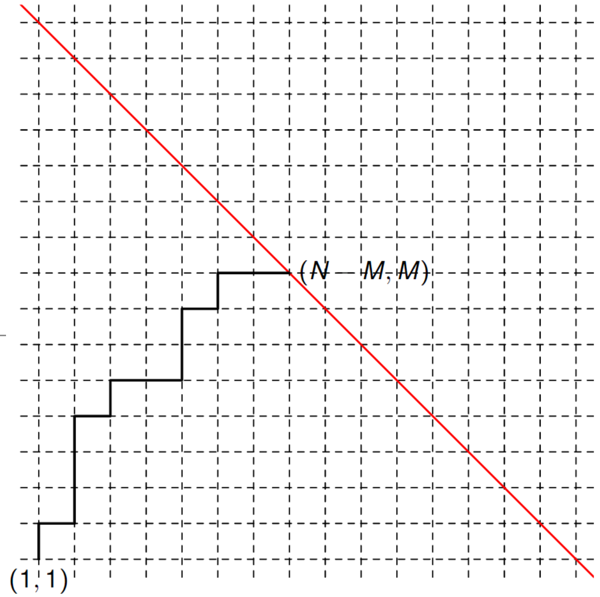
For TASEP: 1 is straightforward

Basic coupling between segment and $\mathbb{Z}$



$F_{N,M}$ = time when there are $\ge M$ numbers $\le M$ to the right of location $N - M$
('truncation operator' in Angel-Holroyd-Romik, 2008)

The strategy: $\max_{1\le M\le N} F^*_{N,M}$



1. Truncation: a segment versus $\mathbb{Z}$
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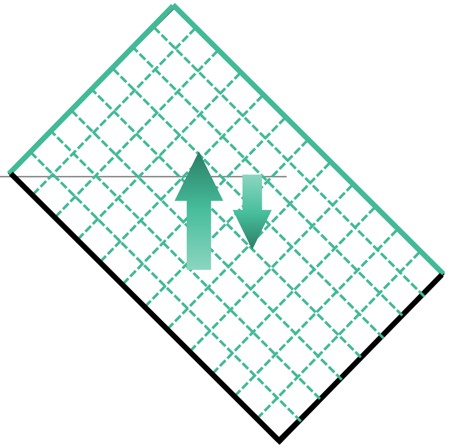
For TASEP: 1 is straightforward

$F_{N,M}$ = time when there are $\ge M$ numbers, from $\le M$ to $> N - M$ (for both segment and $\mathbb{Z}$ models)

By shift-invariance, consider the time when there are $\ge M$ numbers, from ≤ 0 to $> N - 2M$

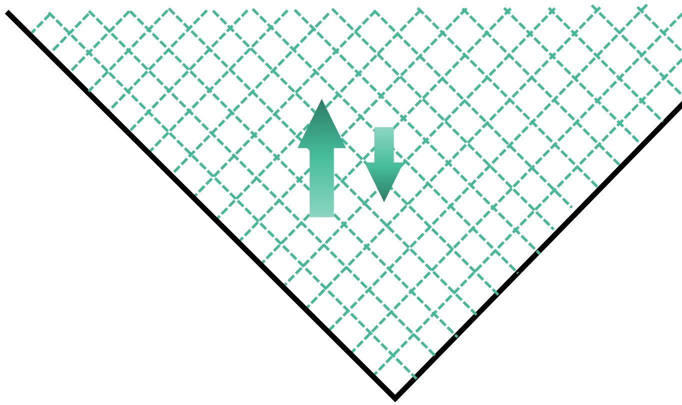
This is on one ordinary step-initial TASEP!

The strategy: $\max_{1 \leq M \leq N} F^*_{N,M}$



1. Truncation: a segment versus $\mathbb{Z}$
2. Shift-invariance Existing from Borodin-Gorin-Wheeler, 2020 or Galashin, 2021
3. Use known LPP/TASEP or ASEP asymptotic results Existing (Quastel-Sarkar, 2020)

For ASEP: 1 is the main task



No exact correspondence (surface up and down)

One side holds: segment surface $\leq \mathbb{Z}$ surface

Idea: should still be roughly the same (suffices to prove the other side)

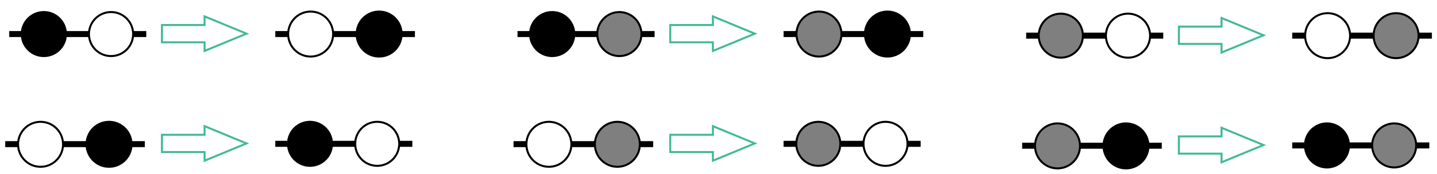
Compare both surfaces

Second-class particles

Particles ● Holes ○ Second-class particles ●

Rate 1

Rate q

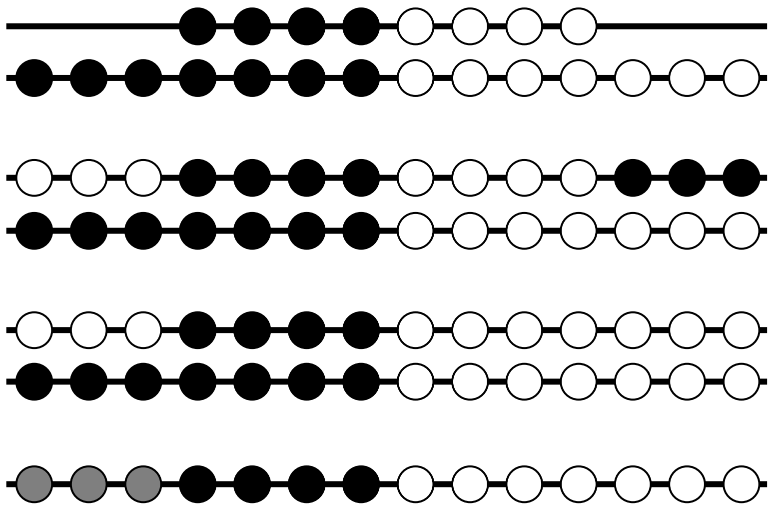


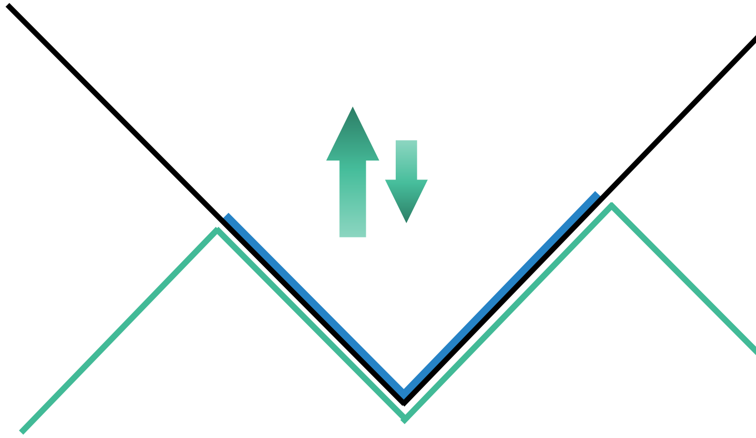
Need to compare

Suffices to compare

A first step

Reduces to





Hecke algebra

S_N embeds into S_N, for $N < N'$

- Basis: $\{T_\omega\}_{\omega \in S_N}$

Also used in Bufetov-Nejjar, 2020 but in a quite different way

- Multiplications: $T_\sigma T_\omega = \begin{cases} T_{\sigma\omega} \\ (1-q)T_\omega + qT_{\sigma\omega} \end{cases}$

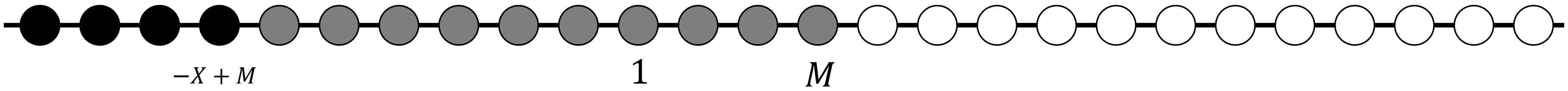
  σ is a nearest neighbor swap;

  depending on whether $\kappa(\sigma\omega) = \kappa(\omega) - 1$ or $\kappa(\sigma\omega) = \kappa(\omega) + 1$

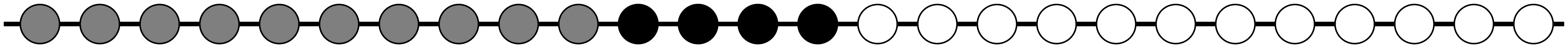
- Each $\sum_{\omega \in S_N} p_\omega T_\omega$ (for $\sum_{\omega \in S_N} p_\omega = 1$) encodes a measure
- ASEP evolution $e^{t\sum_i (\sigma_i - Id)} \to \mathcal{M}_N$ (Mallows element) as $t \to \infty$
- Involution $\mathfrak{i}$: $T_\omega \mapsto T_{\omega^{-1}}$. $\mathfrak{i}(\mathfrak{i}(A)) = A$, $\mathfrak{i}(AB) = \mathfrak{i}(B)\mathfrak{i}(A)$,
  and $\mathfrak{i}\left(e^{t\sum_i(\sigma_i - Id)}\right) = e^{t\sum_i(\sigma_i - Id)}$, $\mathfrak{i}(\mathcal{M}_N) = \mathcal{M}_N$

Use symmetry

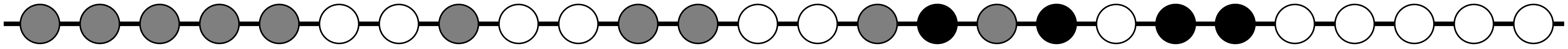
Start from $Id: [-X, X] \to [-X, X]$, apply projection



Apply $\mathcal{M}_{[-X,M]}$ **(initial config of the task)**



Apply $e^{t\sum_{i=-X}^{X}(\sigma_i - Id)}$ **(run for time *t*)**

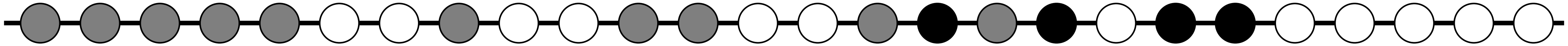


Want: at locations $> N - M$, it is unlikely that

#● is $\ll M$ while #(●+●)$\gg M$

Use $e^{t\sum_{i=-X}^{X}(\sigma_i - Id)}\mathcal{M}_{[-X,M]} = i\left(\mathcal{M}_{[-X,M]}e^{t\sum_{i=-X}^{X}(\sigma_i - Id)}\right)$

Use symmetry

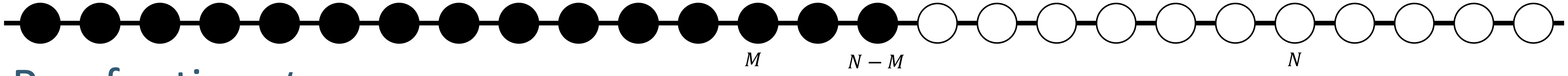


Want: at locations $> N - M$, it is unlikely that

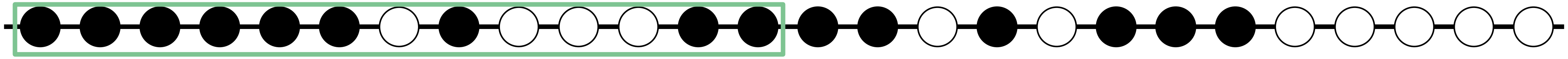
#● is $\ll M$ while #(● + ●) $\gg M$ (under $e^{t\sum_{i=-X}^{X}(\sigma_i - Id)}\mathcal{M}_{[-X,M]}$)

Use $e^{t\sum_{i=-X}^{X}(\sigma_i - Id)}\mathcal{M}_{[-X,M]} = \mathfrak{i}\left(\mathcal{M}_{[-X,M]}e^{t\sum_{i=-X}^{X}(\sigma_i - Id)}\right)$

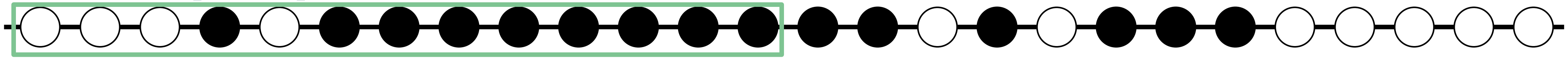
Start from $Id: [-X, X] \to [-X, X]$, apply projection $\leq N - M$ (because of $\mathfrak{i}$)



Run for time t



Apply $\mathcal{M}_{[-X,M]}$

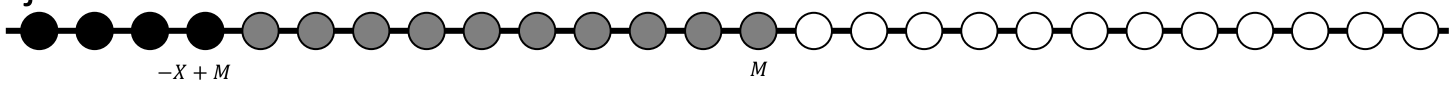


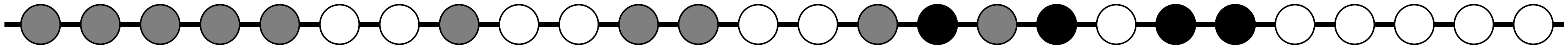
#○ at locations $\leq -X + M$ is $\ll M$ while #○ at locations $\leq M$ is $\gg M$

Use symmetry

$e^{t\sum_{i=-X}^{X}(\sigma_i - Id)}\mathcal{M}_{[-X,M]}$

Projection:



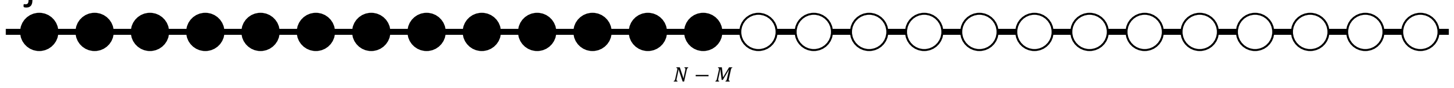


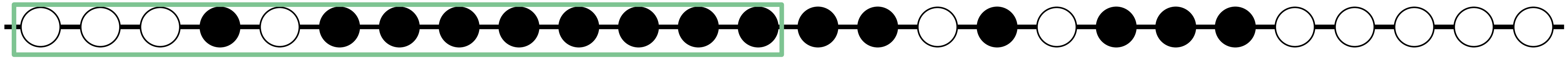
at locations $>N-M$, #● is $\ll M$ while #(●+●) $\gg M$

#(from $\leq -X+M$ to $>N-M$) #(from $\leq M$ to $>N-M$)

$\mathcal{M}_{[-X,M]}e^{t\sum_{i=-X}^{X}(\sigma_i - Id)}$

Projection:



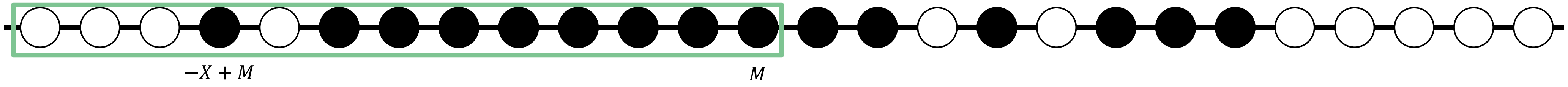


#○ at locations $\leq -X+M$ is $\ll M$ while #○ at locations $\leq M$ is $\gg M$

#(from $>N-M$ to $\leq -X+M$) #(from $>N-M$ to $\leq M$)

Use symmetry

In $\mathcal{M}_{[-X,M]} e^{t\sum_{i=-X}^{X}(\sigma_i - Id)}$



#◯at locations $\leq -X+M$ is $\ll M$ while #◯at locations $\leq M$ is $\gg M$

(exponentially) unlikely by Mallows property!

1. Truncation: a segment versus $\mathbb{Z}$ $F^*_{N,M}$ =Time when ≥M numbers, from ≤ M to >$N-M$
2. Shift-invariance =Time when ≥M numbers, from ≤0 to >$N-2M$
   Existing from Borodin-Gorin-Wheeler, 2020 or Galashin, 2021
3. Use known LPP/TASEP or ASEP asymptotic results $\max_{1\leq M\leq N} F^*_{N,M}$=ASEP point-to-line
   Existing (Quastel-Sarkar, 2020)

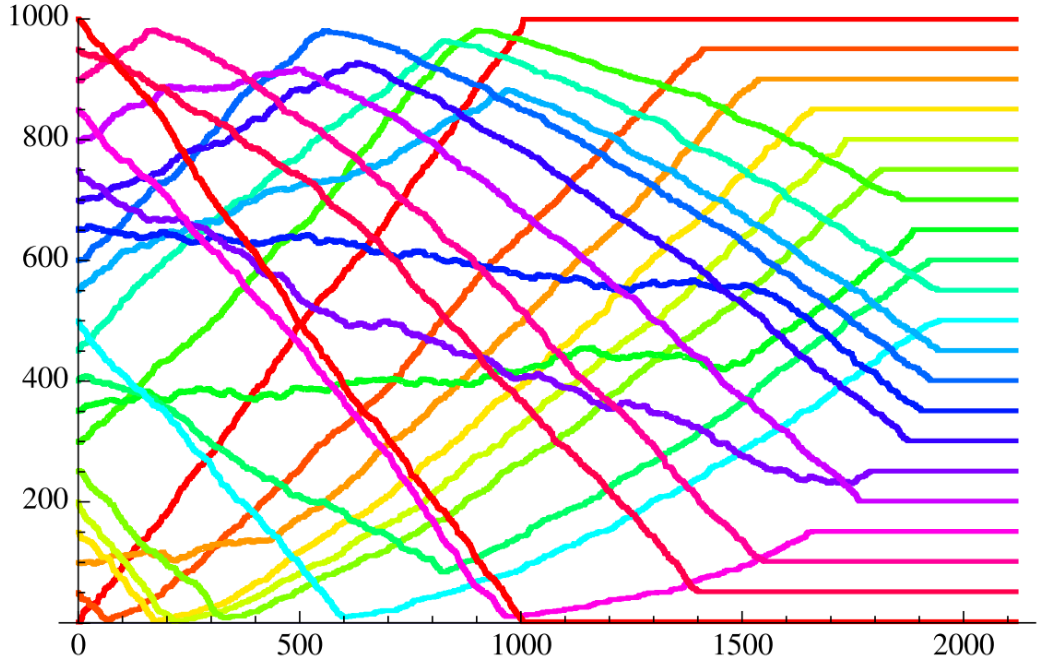
Further questions

Symmetric case? ($q = 1$)

	TASEP absorbing time distribution	**ASEP cutoff profile**	**SSEP cutoff profile**
Ordinary (colorless)	GUE Tracy-Widom (Angel-Holroyd-Romik, 2008)	GUE Tracy-Widom (Bufetov-Nejjar, 2020)	$N^2 \log(N)$ mixing time, N^2Gaussian cutoff (Lacoin, 2015)
Colored	GOE Tracy-Widom (Bufetov-Gorin-Romik, 2020)	GOE Tracy-Widom (Z. 2022)	**Unknown** also called *adjacent card shuffling*; cutoff known (Lacoin, 2013)

Further questions

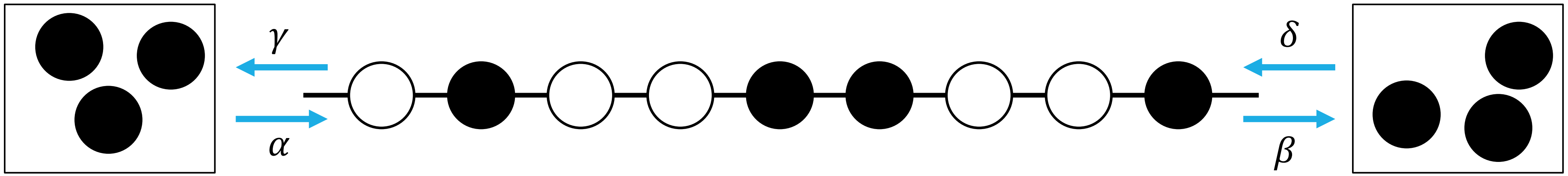
ASEP version of the Oriented Swap Process?



Aggarwal-Corwin-Ghosal, 2022 could be useful

Further questions

ASEP with reservoirs (open boudary)?



Model quantum spin chain

Mixing is largely open!

Thank you!