Factor of IID for the Ising model on the tree

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Joint work with
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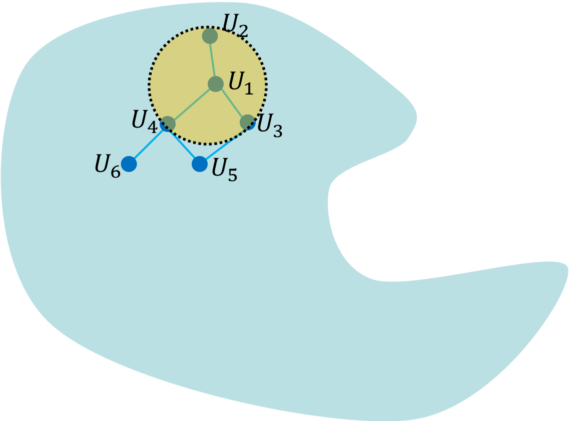
Local Functions

Two perspectives:



Local functions for optimization
Factors of IID – Ergodic Theory

Large Independent sets

Finding large independent sets in d-random regular graphs.

Largest IS is roughly $\frac{(2+o(1))\log d}{d}$n.

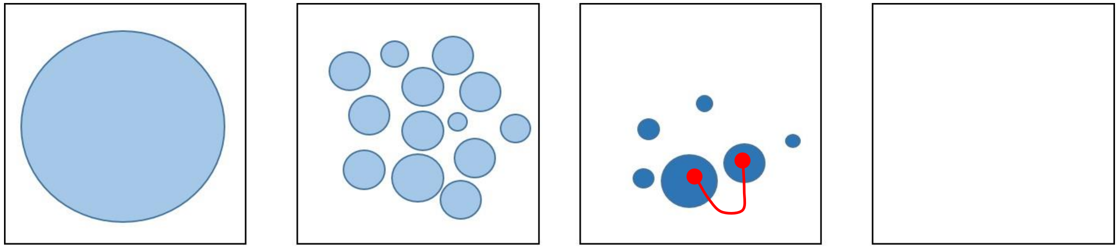
Lauer and Wormald '07 give a local algorithm that finds an IS of size $\frac{(1+o(1))\log d}{d}$n

Iteratively pick vertices with probability p and add them to the set if possible.

Gap of factor of 2.

Large Independent sets

Hatami, Lovasz, and Szegedy asked if there were local algorithms up to $\frac{(2-\epsilon)\log d}{d}$



No for IS up to $\frac{(1+\frac{1}{\sqrt{2}}o(1))\log d}{d}\,n$ Gamarnik, Sudan '14

Independent sets larger than $\frac{(1+\epsilon)\log d}{d}n$ come in well separated clusters.

No for IS larger than $\frac{(1+\epsilon)\log d}{d}n$ Rahman, Virag '17

Factors of IID

Goal: reconstruct $\sigma: V \to X$ e.g. colouring, matching, Ising from IID random variables $\{U_x\}_{x \in V}$.

On a transitive graph e.g. $\mathbb{Z}^d, \mathbb{T}^d$ with randomness a FIID is a measurable function

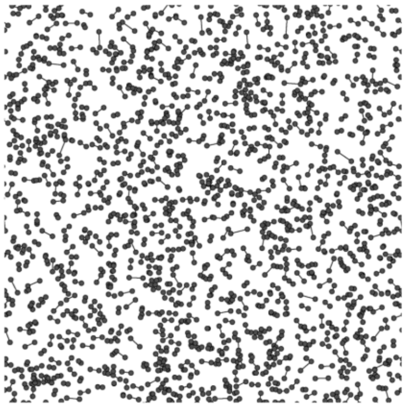
$$f: [0,1]^V \to X, \qquad \sigma(x) = f\big(\tau_x\{U_y\}\big),$$

where τ_x is the shift operator $(\tau_x\{U\})_z = U_{z-x}$.

Note that there is no assumption on the radius but by measurability it can be approximated by bounded radius.

On $\mathbb{Z}^d$ being a factor of IID is equivalent to being isomorphic to a Bernoulli shift.

Factors of IID



Matchings

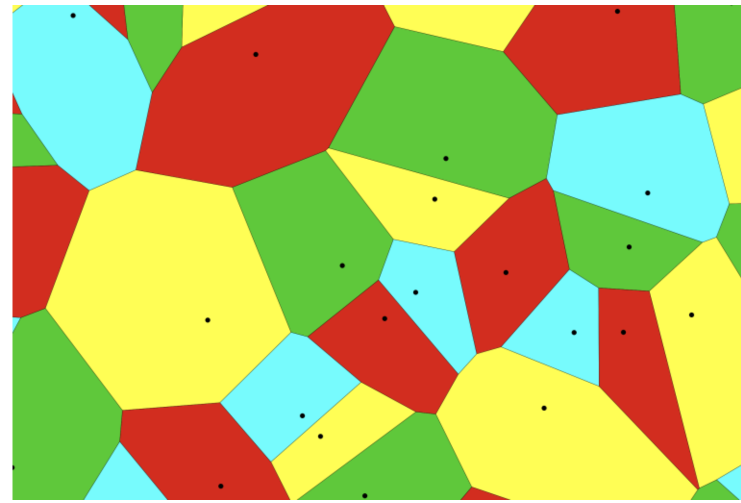
Holroyd, Pemantle, Peres, Schramm '09

Non-amenable graphs - Lyons Nazarov '11

Gaussian Wave function FIID

Thresholding leads to density 0.43 IS on 3-regular tree

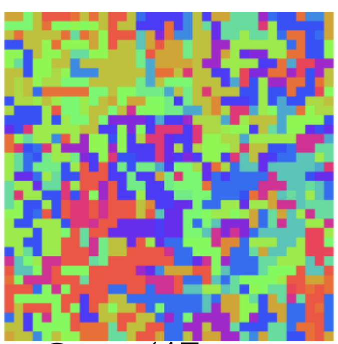
Csóka, Gerencsér, Harangi, Virág '15



Colourings of Planar Graphs

Angel, Benjamini, Gurel-Gurevich, Meyerovitch, Peled '12

Timar '11



Swart '17

Divide and Colour

Partition vertices and colour components independently e.g. Ising, Potts, Voter

Voter stationary distribution S., Zhang '19

Gibbs measures with spatial mixing

Spinka '20.

Ising model on trees (Free measure)

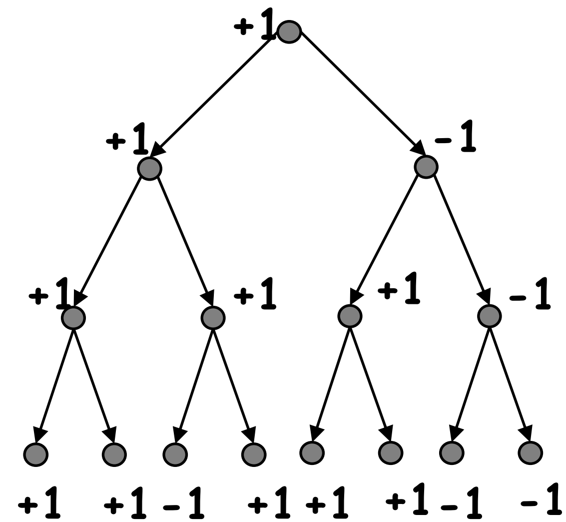
A random assignment
$\sigma \in \{-1,+1\}^V$
with distribution

$$\mathbb{P}[\sigma] = \frac{1}{Z}\exp(\beta \sum_{u \sim v} \sigma_u \sigma_v)$$

Alternatively: a broadcast model where a vertex is equal to its parent with probability

$$\frac{1}{2} + \frac{1}{2}\tanh\beta$$

$$Cov(\sigma_u, \sigma_v) = (\tanh\beta)^{d(u,v)}$$



FK model: $\xi \in \{0,1\}^E$

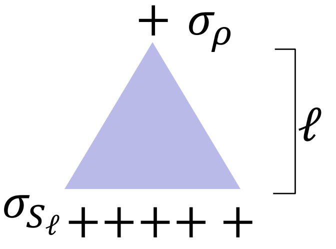
$$\mathbb{P}[\xi] = \frac{1}{Z} y^{\Sigma \xi_u} 2^{\# C(\xi)}$$

where $C(\xi)$ is number of connected components.
On tree percolation w.p.

$$p = \tanh\beta$$

Phase Transitions (Uniqueness)

Uniqueness Threshold: $\tanh \beta = d^{-1}$



The critical value for a distant boundary to effect the root

$$\lim_{\ell} \mathbb{P}\left[\, \sigma_\rho = + \middle| \, \sigma_{S_\ell} \equiv + \right] = 1/2 \quad \Leftrightarrow \tanh \beta \le d^{-1}$$

For larger β there exist multiple *Gibbs measures* (extensions to infinite graph) such as the *plus measure*.

High Temperature: $\tanh \beta \le d^{-1}$

FK – model $p \le d^{-1}$ so all components are finite.

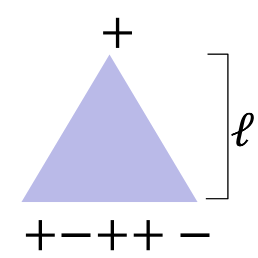
There exists a FIID.

Phase Transitions (Reconstruction)

Reconstruction/Extremeality Threshold: $\tanh\beta = d^{-1/2}$

Critical value for distant vertices to affect the root.

$$\lim_{\ell} \mathbb{P}\left[\sigma_\rho = + \middle| \sigma_{S_\ell}\right] = 1/2 \; a.s. \iff \tanh\beta \le d^{-1/2}$$



Low Temperature $\tanh\beta > d^{-1/2}$

Distant spins contain information about the root,

$$Var\left((d\tanh\beta)^{-\ell}\Sigma_{u\in S_\ell}\,\sigma_u\right) \to C$$

$$\lim_{\ell} Cov\left(\sigma_\rho, (d\tanh\beta)^{-\ell}\Sigma_{u\in S_\ell}\,\sigma_u\right) > 0$$

Is FIID possible with such long range dependencies?

No FIID for low temperature.

Suppose $\sigma_x = f(\tau_x(\{U\}))$

There exists a finite range factor g such that

$$\sigma'_x = g(\tau_x(\{U\})) \in \{-1,1\}, \qquad \mathbb{P}[\sigma'_x \neq \sigma_x] \leq \epsilon, \qquad \mathbb{E}[\sigma'_x] = 0$$

Then we have

$$\lim_{\ell} Cov\left(\sigma'_\rho, (d\tanh\beta)^{-\ell} \Sigma_{u \in S_\ell}\, \sigma_u\right) > 0$$

By symmetry

$$Cov\left(\sigma'_\rho, (d\tanh\beta)^{-\ell} \Sigma_{u \in S_\ell}\, \sigma_u\right) = Cov\left(\sigma_\rho, (d\tanh\beta)^{-\ell} \Sigma_{u \in S_\ell}\, \sigma'_u\right)$$

But

$$Var\left((d\tanh\beta)^{-\ell} \Sigma_{u \in S_\ell}\, \sigma'_u\right) \leq (d\tanh\beta)^{-2\ell} * Cd^\ell \to 0$$

since σ'_u are uncorrelated at large distances.

Intermediate temperatures?

Lyons '14 asked, when $\tanh\beta \in (d^{-1}, d^{-1/2})$ is there a factor of IID?

Attempt 1: In the FK model, the components are infinite – no translation invariant way to assign the colours.

Attempt 2: Peres suggest the following:

Construct σ using the Glauber Dynamics Markov chain.

Each vertex has rate 1 Poisson clock

Update it according to stationary distribution.

Use coupling from the past i.e. run for $t \in (-\infty..0]$

Different initial condition such as $+$ lead to different Gibbs measures.

Suggests IID initial configuration.

Simulations/heuristics suggest it does not converge almost surely.

Intermediate temperatures?

Attempt 3: Assign the vertices in order.

There's no T.I. ordering of all the vertices.

But we can assign them times $T_v \in [0,1]$ IID plus $U_v \in [0,1]$

Set $\sigma_v = 1$ if $U_v \leq \mathbb{P}[\sigma_v = 1 \mid \{\sigma_u\}_{u:T_u<T_v}]$

Problem: There exist multiple solution given $\{U_v, T_v\}_{v\in V}$.
Difficult to control the effect of far away choices.

Attempt 4: Reveal noisy version of σ_v, $H_{v,1}, H_{v,2}, \ldots$ at times $T_{v,i}$ where $\mathbb{P}\left[\sigma_v = H_{v,i}\right] = 1/2 + \alpha$.

Then $\mathrm{X_v(n)} := \Sigma_{i=1}^{n} H_{v,i} \approx N(2\alpha\, \sigma_v\, n, n)$.

Still requires hard choices - idea take $\alpha \to 0$. Asymptotically $\mathrm{X_v(n)}$ is Brownian motion with drift.

FIID Construction

We will build a process $X_t(v) = \sigma_v t + B_t(v)$ where $B_t(v)$ are independent Brownian motions.

Easy to construct if we already know σ_v (but we don't).

Easier example: single vertex v. Then if $\mathcal{F}_t$ is the filtration generated by X_t then

$$\mathbb{E}[\,\sigma_v \mid \mathcal{F}_t\,] = \tanh X_t(v)$$

we have can construct it by

$$dX_t(v) = \tanh X_t(v)\,dt + dB_t(v)$$

The stochastic differential equation has a unique strong solution, that is we can construct $X_t(v)$ given $B_t(v)$.

FIID Construction

For all v we want to construct $X_v(t)$ simultaneously.

The Ising model with external field $\{h_v\}$ is given by

$$\mathbb{P}_h[\sigma] = \frac{1}{Z}\exp(\beta \sum_{u\sim v} \sigma_u\sigma_v + \sum_v h_v\sigma_v)$$

Then by Bayes rule with $\mathcal{F}_t = \{X_s(v)\}_{s\leq t, v\in T}$

$$\mathbb{E}[\,\sigma_v \mid \mathcal{F}_t\,] = \mathbb{E}_{X_t}[\sigma_v]$$

And $X_t(v)$ is a solution of

$$dX_t(v) = \mathbb{E}_{X_t}[\sigma_v]\mathrm{dt} + dB_t(v)$$

This is an infinite dimensional SDE.

It has multiple strong solutions.

<u>Theorem</u> (Nam, S., Z.) When $\tanh\beta \in \left(d^{-1}, \delta d^{-\frac{1}{2}}\right)$ there is a strong solution that gives a FIID for the free Ising model.

FIID Construction

For the infinite dimensional SDE

$$dX_t(v) = \mathbb{E}_{X_t}[\sigma_v]\mathrm{dt} + dB_t(v)$$

We now construct a strong solution that is translation invariant; i.e., define a translation invariant function $\mathcal{F}: B \mapsto X$.

A step back: on a finite graph, this SDE has a unique strong solution. On a ball of radius R around the root ρ (T_R) we build the SDE

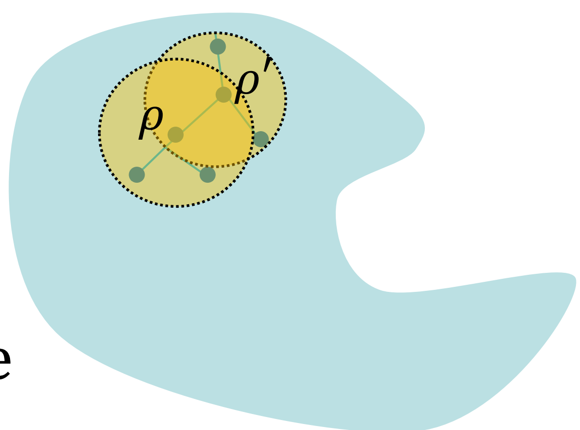
$$dX_t^R(v) = \mathbb{E}_{X_t^R}[\sigma_v]\mathrm{dt} + dB_t(v) \qquad \forall v \in T_R$$

<u>Theorem</u> (Nam, S., Z.)

When $\tanh\beta \in \left(d^{-1}, \delta d^{-\frac{1}{2}}\right)$, almost surely as $R \to \infty$

$$X_t^R(v) \to X_t(v)$$

And the limit $X_t(v)$ is independent of the choice of ρ.



Comparing X_t^R and X_t^{R+1}

To show convergence, bound the difference between X_t^R and X_t^{R+1}.
Again we take a continuous approach.

For $\gamma \in [0, \beta]$ define

$$\mathbb{P}_{h,\gamma}[\sigma] = \frac{1}{Z}\exp(\beta \sum_{\substack{u\sim v \\ u,v\in T_R}} \sigma_u\sigma_v + \gamma \sum_{\substack{u\sim v \\ u\in T_R, v\in T_{R+1}}} \sigma_u\sigma_v + \sum_{v\in T_{R+1}} h_v\sigma_v)$$

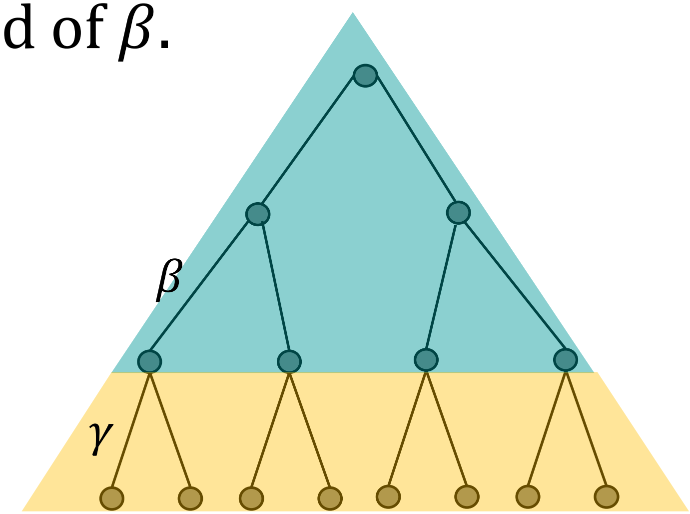
In words, it is Ising model on T_{R+1} with external field h ($\mathbb{P}_h$), and the inverse temperature on leaves is γ instead of β.

Let $X_t^{R,\gamma}$ be the solution of

$$dX_t^{R,\gamma}(v) = \mathbb{E}_{X_t^{R,\gamma},\gamma}[\sigma_v]\mathrm{dt} + dB_t(v)$$

By varying γ we interpolate:

$X_t^{R,0} = X_t^R$ and $X_t^{R,\beta} = X_t^{R+1}$



Comparing X_t^R and X_t^{R+1}

Denote $H_t^{R,\gamma} = \frac{d}{d\gamma} X_t^{R,\gamma}$.

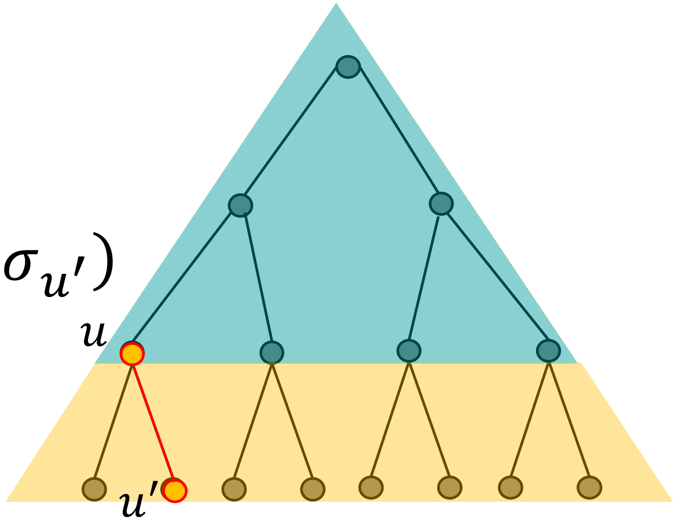
From $dX_t^{R,\gamma}(v) = \mathbb{E}_{X_t^{R,\gamma},\gamma}[\sigma_v]\mathrm{dt} + dB_t(v)$ we compute that

$\frac{d}{dt} H_t^{R,\gamma}(v) = \partial_\gamma \mathbb{E}_{X_t^{R,\gamma},\gamma}[\sigma_v] = M_t H_t^{R,\gamma}(v) + N_t(v)$

where M_t is a $T_{R+1} \times T_{R+1}$ matrix:

$M_t(u,v) = Cov_{X_t^{R,\gamma}}(\sigma_u, \sigma_v)$, and

$N_t(v) = \sum_{u \in T_R, u \sim u', u' \in \partial T_{R+1}} Cov_{X_t^{R,\gamma}}(\sigma_v, \sigma_u \sigma_{u'})$



Thus we can write

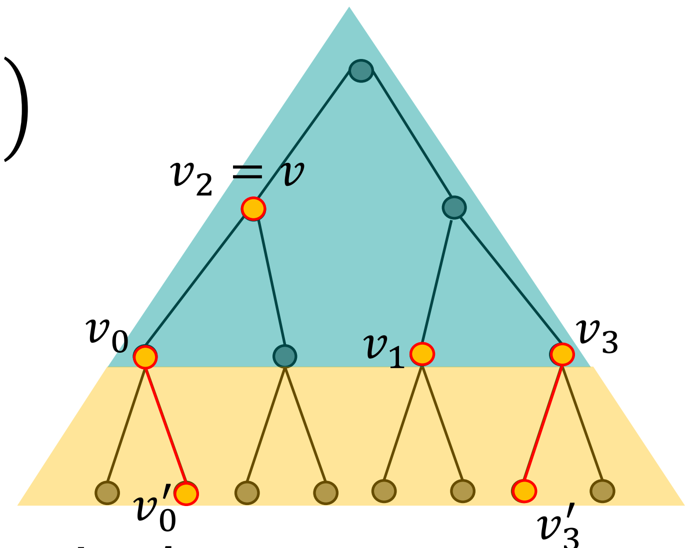
$$H_t^{R,\gamma} = \sum_{k=1}^{\infty} \int_{0<t_1<\cdots<t_k<t} M_{t_k} \dots M_{t_2} N_{t_1}\, dt_1 \dots dt_k$$

Comparing X_t^R and X_t^{R+1}

To bound $X_t^R - X_t^{R+1}$, we study the second moment $\mathbb{E}[(H_t(v))^2]$.

From $H_t = \sum_{k=1}^{\infty} \int_{0<t_1<\cdots<t_k<t} M_{t_k} \ldots M_{t_2} N_{t_1}\, dt_1 \ldots dt_k$,
we can write $(H_t(v))^2$ as sum and integral of terms like

$$Cov_{X_{t_0}}\left(\sigma_{v_1}, \sigma_{v_0}\sigma_{v_0'}\right) Cov_{X_{t_k}}\left(\sigma_{v_k}, \sigma_{v_{k+1}}\sigma_{v_{k+1}'}\right) \times \prod_{i=1}^{k-1} Cov_{X_{t_i}}(\sigma_{v_i}, \sigma_{v_{i+1}})$$



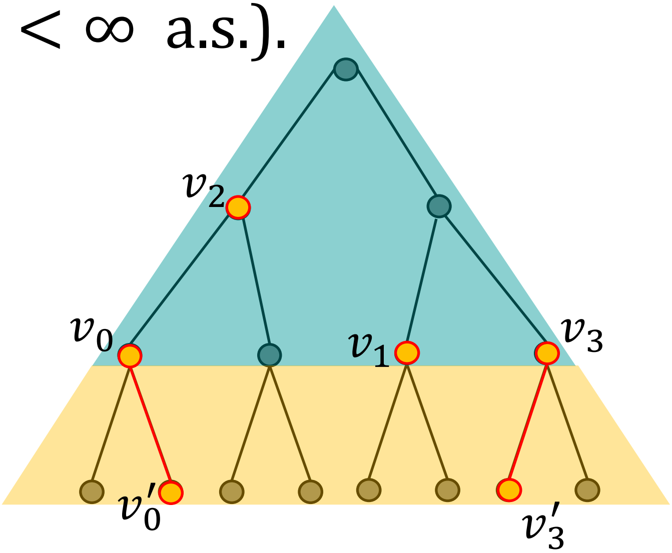
where $v \in \{v_0, \ldots, v_{k+1}\}$,
and $v_0, v_{k+1} \in T_R, v_0 \sim v_0', v_{k+1} \sim v_{k+1}'$, and $v_0', v_{k+1}' \in \partial T_{R+1}$

Comparing X_t^R and X_t^{R+1}

$$Cov_{X_{t_0}}\left(\sigma_{v_1}, \sigma_{v_0}\sigma_{v_0'}\right) Cov_{X_{t_k}}\left(\sigma_{v_k}, \sigma_{v_{k+1}}\sigma_{v_{k+1}'}\right) \prod_{i=1}^{k-1} Cov_{X_{t_i}}\left(\sigma_{v_i}, \sigma_{v_{i+1}}\right)$$

where $v \in \{v_0, \ldots, v_{k+1}\}$, and $v_0, v_{k+1} \in T_R, v_0 \sim v_0', v_{k+1} \sim v_{k+1}'$, and $v_0', v_{k+1}' \in \partial T_{R+1}$

Given v, we wish the sum (and integral) of all such terms decay fast in R (want $\sum_R \left| X_t^R(v) - X_t^{R+1}(v) \right| < \infty$ a.s.).



A direct bound: in a tree, external fields only decrease covariances:

$$0 \le Cov_h(\sigma_v, \sigma_u) \le Cov_0(\sigma_v, \sigma_u)$$
$$= (\tanh \beta)^{\mathrm{dist}(u,v)} < d^{-\mathrm{dist}(u,v)/2}$$

(recall $\tanh \beta < \delta d^{-\frac{1}{2}}$).

A bound of $(\tanh \beta)^{\mathrm{dist}(v_0,v_1)+\cdots+\mathrm{dist}(v_k,v_{k+1})}$ is not enough!

e.g. $\sum_{v_0,v_2 \in \partial T_R} d^{-(\mathrm{dist}(v_0,v)+\mathrm{dist}(v,v_2))/2} \approx d^R$.

Comparing X_t^R and X_t^{R+1}

$$Cov_{X_{t_0}}\left(\sigma_{v_1}, \sigma_{v_0}\sigma_{v_0'}\right) Cov_{X_{t_k}}\left(\sigma_{v_k}, \sigma_{v_{k+1}}\sigma_{v_{k+1}'}\right) \prod_{i=1}^{k-1} Cov_{X_{t_i}}(\sigma_{v_i}, \sigma_{v_{i+1}})$$

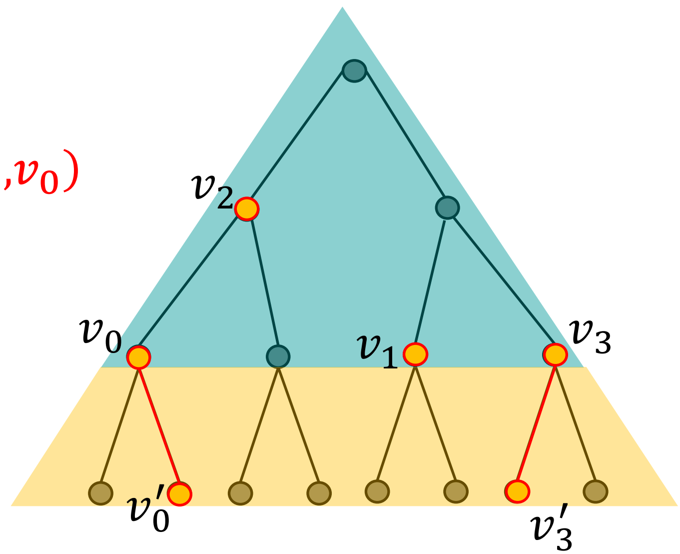
where $v \in \{v_0, \dots, v_{k+1}\}$, and $v_0, v_{k+1} \in T_R$, $v_0 \sim v_0'$, $v_{k+1} \sim v_{k+1}'$, and $v_0', v_{k+1}' \in \partial T_{R+1}$

A bound of $(\tanh\beta)^{\text{dist}(v_0,v_1)+\cdots+\text{dist}(v_k,v_{k+1})}$ is not enough!

e.g. $\sum_{v_0,v_2\in\partial T_{R+1}} d^{-(\text{dist}(v_0,v)+\text{dist}(v,v_2))/2} \approx d^R$.

Suffices to have one extra factor:

$\frac{1}{k!}(\tanh\beta)^{\text{dist}(v_0,v_1)+\cdots+\text{dist}(v_k,v_{k+1})+\text{dist}(v_{k+1},v_0)}$



Each $(\tanh\beta)^L$ corresponds to a walk starting and ending at v with length L; there are $\approx (d + o(1))^{L/2}$ such walks.

Or the prob of a random walk starting from v. At each step, with prob ½ moves farther from v, and prob ½ moves closer to v.

Estimating $\mathbb{E}\left[\left(H_t^{R,\gamma}(v)\right)^2\right]$

$$\mathbb{E}\left[Cov_{X_{t_0}^{R,\gamma}}\left(\sigma_{v_1},\sigma_{v_0}\sigma_{v_0'}\right)Cov_{X_{t_k}^{R,\gamma}}\left(\sigma_{v_k},\sigma_{v_{k+1}}\sigma_{v_{k+1}'}\right)\prod_{i=1}^{k-1}Cov_{X_{t_i}^{R,\gamma}}(\sigma_{v_i},\sigma_{v_{i+1}})\right]$$

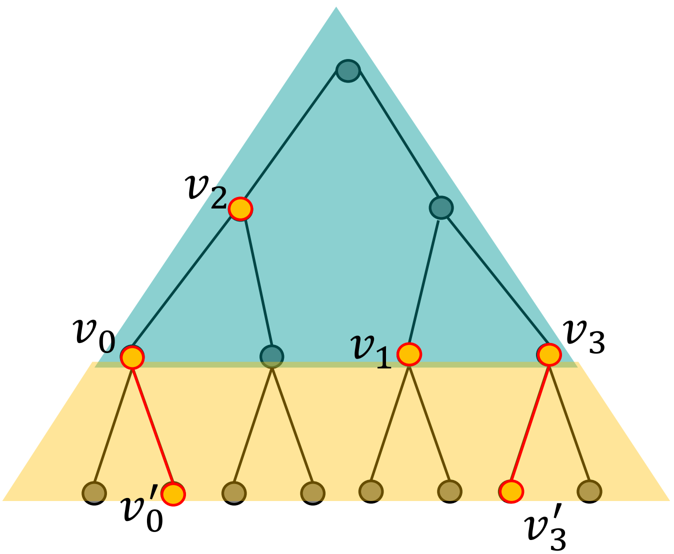
$$< (C\tanh\beta)^{\mathrm{dist}(v_0,v_1)+\cdots+\mathrm{dist}(v_k,v_{k+1})+\mathrm{dist}(v_{k+1},v_0)}$$

Actually we can write

$$Cov_{X_t}(\sigma_v,\sigma_u\sigma_{u'}) = \frac{\sinh\left(X_t(u')\right)(\tanh\beta)^{\mathrm{dist}(u',v)}}{2\left(\bar{Z}_{X_t}(u',v)\right)^2}$$

$$Cov_{X_t}(\sigma_u,\sigma_v) = \frac{(\tanh\beta)^{\mathrm{dist}(u,v)}}{\left(\bar{Z}_{X_t}(u,v)\right)^2}$$

where $\bar{Z}_{X_t}(u,v) = \frac{Z_{X_t}(u,v)}{Z_0(u,v)} \geq 1.$



Need: $\mathbb{E}\left[\frac{\sinh\left(X_{t_0}(v_0)\right)\sinh\left(X_{t_k}(v_{k+1})\right)}{\left(\Pi_i\bar{Z}_{X_{t_i}}(v_i,v_{i+1})\right)^2}\right] < (C\tanh\beta)^{\mathrm{dist}(v_{k+1},v_0)}$

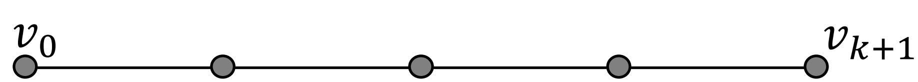
Estimating $\mathbb{E}\left[\left(H_t^{R,\gamma}(v)\right)^2\right]$

Need: $\mathbb{E}\left[\frac{\sinh\left(X_{t_0}(v_0)\right)\sinh\left(X_{t_k}(v_{k+1})\right)}{\left(\Pi_i \bar{Z}_{X_{t_i}}(v_i,v_{i+1})\right)^2}\right] < (C\tanh\beta)^{\mathrm{dist}(v_{k+1},v_0)}$

The LHS is 'like' $\mathbb{E}\left[\sigma_{v_0}\sigma_{v_{k+1}}\right]$, which equals $(\tanh\beta)^{\mathrm{dist}(v_{k+1},v_0)}$.

However, the weights make it hard to compute directly.

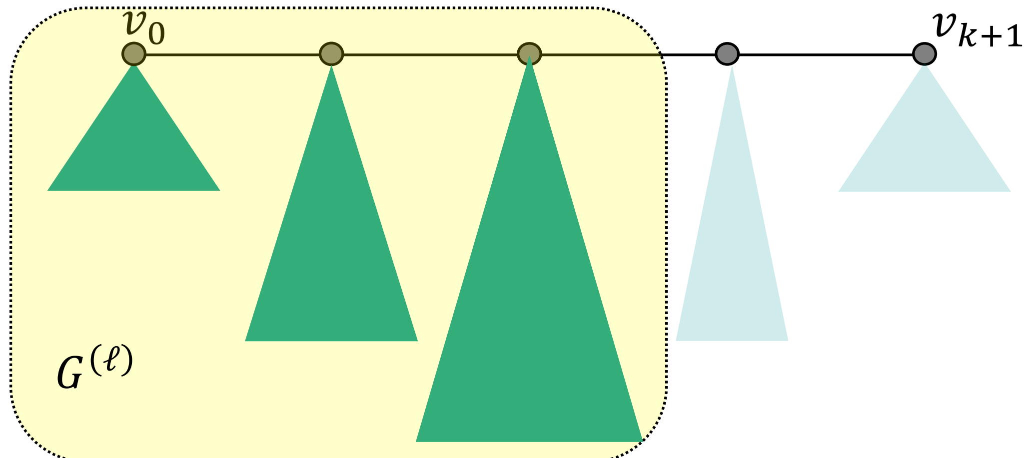
Recall how we compute $\mathbb{E}\left[\sigma_{v_0}\sigma_{v_{k+1}}\right]$: one way is to take the path from v_0 to v_{k+1}, and it is a Markov chain.



Solution: reveal the field gradually

Estimating $\mathbb{E}\left[\left(H_t^{R,\gamma}(v)\right)^2\right]$

$$\mathbb{E}\left[\frac{\sinh\left(X_{t_0}(v_0)\right)\sinh\left(X_{t_k}(v_{k+1})\right)}{\left(\Pi_i \bar{Z}_{X_{t_i}}(v_i, v_{i+1})\right)^2}\right]$$



For $0 \le \ell \le \mathrm{dist}(v_{k+1}, v_0)$, construct measure $\mu_{\pm,\ell}$:

$$d\mu_{\pm,\ell} = \frac{I(\pm X_{t_0}(v_0) > 0)\sinh\left(\left|X_{t_0}(v_0)\right|\right)}{\left(\Pi_i \bar{Z}^{(\ell)}_{X_{t_i}}(v_i, v_{i+1})\right)^2} d\mu$$

Where $\bar{Z}^{(\ell)}_{X_{t_i}}$ is $\bar{Z}_{X_{t_i}}$ restricted to $G^{(\ell)}$.

We couple $\mu_{+,\ell}$ with $\mu_{-,\ell}$ inductively (in ℓ), minimizing

$$\mathbb{E}_{\mu_{+,\ell}}\left[\sinh\left(X_{t_k}(v_{k+1})\right)\right] - \mathbb{E}_{\mu_{-,\ell}}\left[\sinh\left(X_{t_k}(v_{k+1})\right)\right]$$

Open Problems

Some directly related questions:

1) Extend the analysis to full intermediate regime?

   The reason we require $\tanh \beta \in (d^{-1}, \delta d^{-1/2})$ instead of $\tanh \beta \in (d^{-1}, d^{-1/2})$ is technical, rather than intrinsic.

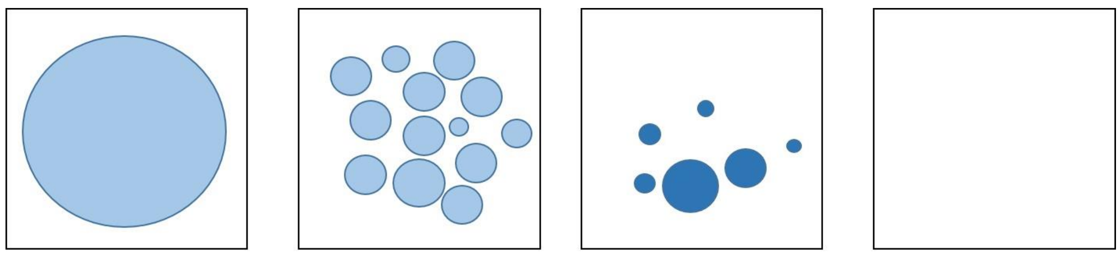
2) Find a simpler FIID?

   Is there a more direct construction, avoiding the computations?

3) What is the relationship between FIIDs and reconstruction/extremality?

1RSB models

Several models (colourings, large independent sets, k-sat) are in the one step replica symmetry breaking universality class.



Thresholds	Colouring	IS/Hardcore
Clustering	$d \log d$	$\alpha = \frac{\log d}{d}$
Algorithms	$d \log d$	$\alpha = \frac{\log d}{d}$
Colourability/ MAX IS	$2d \log d$	$\alpha = \frac{2\log d}{d}$

Full RSB models

Example: Sherrington-Kirkpatrick model, antiferromagnetic Ising model.

For spin glasses Subag '18, Montanari '19, El Alaoui, Montanari Selke '20 gave algorithms that give $(1-\epsilon)$ approximation to the ground state.

Should also apply to anti-ferromagnetic Ising model:

$$\text{Max Cut} = \text{n}\left(\frac{d}{2} + \sqrt{d}\, P_* + o\big(\sqrt{d}\big)\right)$$

[Dembo, Montanari, Sen]

The Gibbs measure is not locally optimal.

Thank you for listening