Scaling limit of half-space last-passage percolation models

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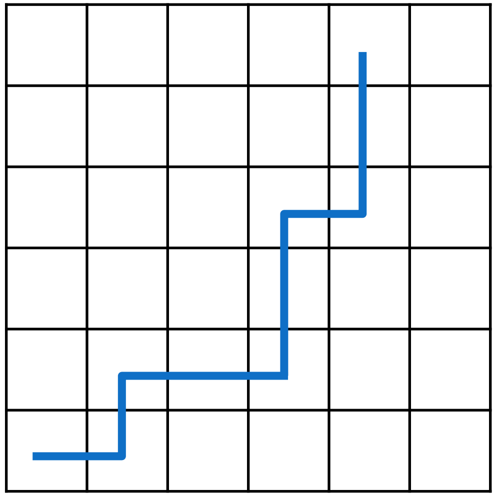
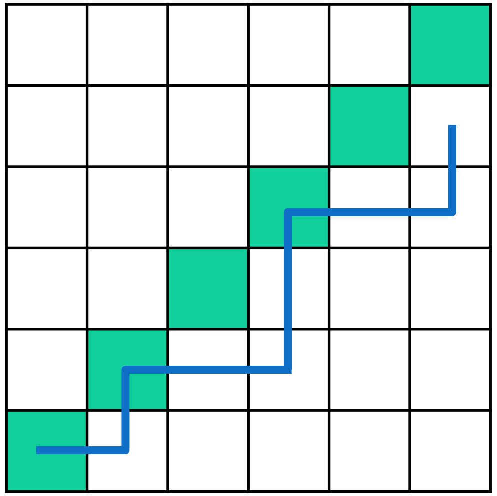
Joint work with Duncan Dauvergne

Exponential Last Passage Percolation



Full-space

- $\xi(v) \sim \mathrm{Exp}(1)$, for $v \in \mathbb{Z}^2$
- $T_{u,v} = \max_{\gamma} \sum_{v \in \gamma} \xi(v)$



Half-space / symmetrized

- $\xi(i,j) = \xi(j,i) \sim \mathrm{Exp}(1)$, for $i > j \in \mathbb{Z}$
- $\xi(i,i) \sim \mathrm{Exp}(a)$, for $i \in \mathbb{Z}$
- $T^a_{u,v} = \max_{\gamma} \sum_{v \in \gamma} \xi(v)$

For both u, v on or below diagonal: restricted path suffices

Scaling limit in full-space

Exact-solvable model

(Rost, 81') $(4n)^{-1}T_{(1,1),(n,n)} \to 1$ in probability

(Johannson, 00') Using RSK correspondence, i.e., LPP realization → pair of Young tableaux

$$\mathbb{P}\left(T_{(1,1),(n,n)} < t\right) = \frac{1}{Z}\int_{[0,t]^n}\prod_{1\le i<j\le n}\left(x_i - x_j\right)^2\prod_{1\le i\le n} e^{-x_i}\,\mathrm{d}x_i$$

Then (using Fredholm determinant)

$$\mathbb{P}\left(T_{(1,1),(n,n)} < 4n + x2^{4/3}n^{1/3}\right) \to F_{GUE}(x)$$

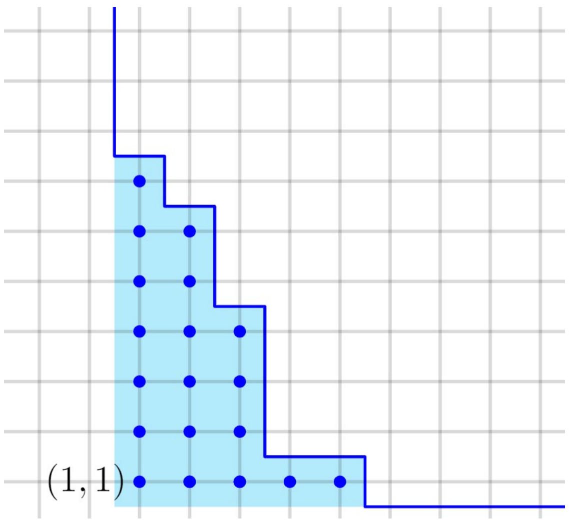
i.e., $T_{(1,1),(n,n)} \approx 4n + 2^{4/3}n^{1/3}X$*, with* X *~GUE Tracy-Widom*

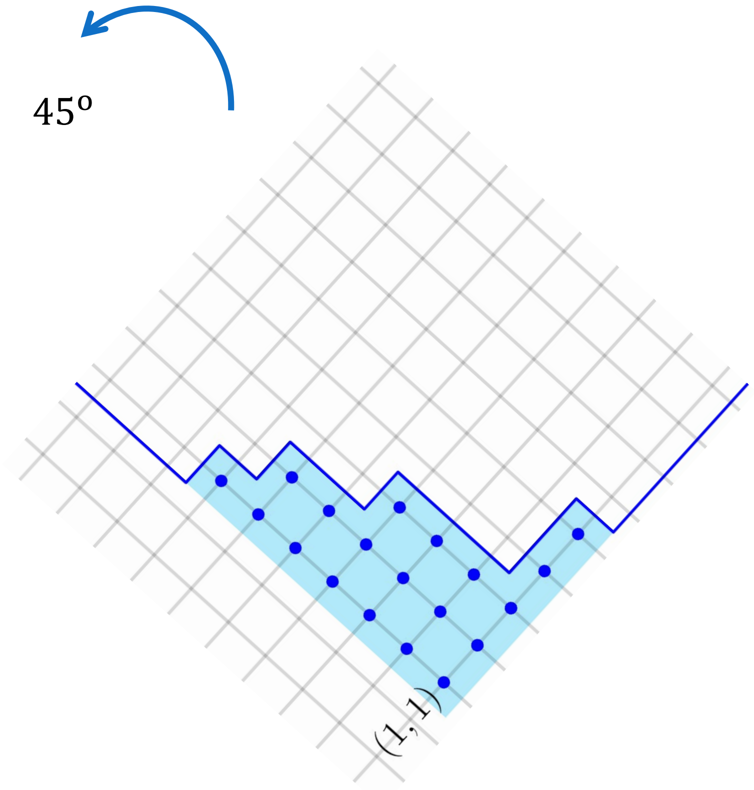
(Borodin-Ferrari, 08') $2^{-4/3}n^{-1/3}\left(T_{(1,1),(n+x(2n)^{2/3},n-x(2n)^{2/3})} - 4n\right) \to \mathcal{A}(x)$

$\mathcal{A}$: Airy process; $\mathcal{A}(x) + x^2$ is stationary

As a growth process

Metric ball $B_t = \{v \in \mathbb{N}^2 : T_{(1,1),v} < t\}$





Height function $H_t : \mathbb{Z} \to \mathbb{Z}$,
with $H_t(x+1) = H_t(x) \pm 1$,

- Markov process : flip with rate 1 independently
- Metric ball B_t: initial condition $H_0(x) = |x|$
- Can also take more general initial H_0

Scaling limit as a Markov process?

KPZ fixed point (full-space)

Height function process $(H_t)_{t\geq 0}$, starting from general H_0

For a family $\left(H_t^{(\varepsilon)}\right)_{t\geq 0}$, scaling: $h_t^{\varepsilon}(x) = -\varepsilon^{1/2}\left(H^{(\varepsilon)}_{2\varepsilon^{-3/2}t}(2\varepsilon^{-1}x) - \varepsilon^{-3/2}t\right)$

(Matetski-Quastel-Remenik, 16')

As $\varepsilon \to 0$, assume that h_0^{ε} converges to $f: \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$. Then for $t > 0$, h_t^{ε} converges as $\varepsilon \to 0$, to $\mathfrak{h}_t f$, the **KPZ fixed point** starting from f at time t.

KPZ fixed point is a *Markov process on (upper semi-continuous) functions* $\mathbb{R} \to \mathbb{R} \cup \{-\infty\}$.

Exact formulas For upper semi-continuous $f: \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$
and lower semi-continuous $g: \mathbb{R} \to \mathbb{R} \cup \{\infty\}$, there is

$$\mathbb{P}(\mathfrak{h}_t f < g) = \det\left(\mathbf{I} - \mathbf{K}_{t/2}^{\mathrm{hypo}(f)} \mathbf{K}_{-t/2}^{\mathrm{epi}(g)}\right)_{L^2(\mathbb{R})}$$

In particular, $\mathfrak{h}_1 \delta_0 = \mathcal{A}$.
Here $\delta_0(0) = 0$, $\delta_0(x) = -\infty$ for any $x \neq 0$.

Ganguly, S. Random metric geometries on the plane and Kardar-Parisi-Zhang universality. *AMS Notices (Jan 2022)*

Directed landscape (full-space)

(Dauvergne-Ortmann-Virag, 18', Dauvergne-Virag, 21') 2D 1:2:3 scaling limit

As $n \to \infty$

$(x, s; y, t) \mapsto 2^{-4/3} n^{-1/3} \left(T_{R_n(x,s), R_n(y,t)} - 4n(t-s) - 2^{8/3} n^{2/3}(y-x)\right)$ converges

where $R_n(x, s) = (ns + 2^{5/3} n^{2/3} x, ns)$

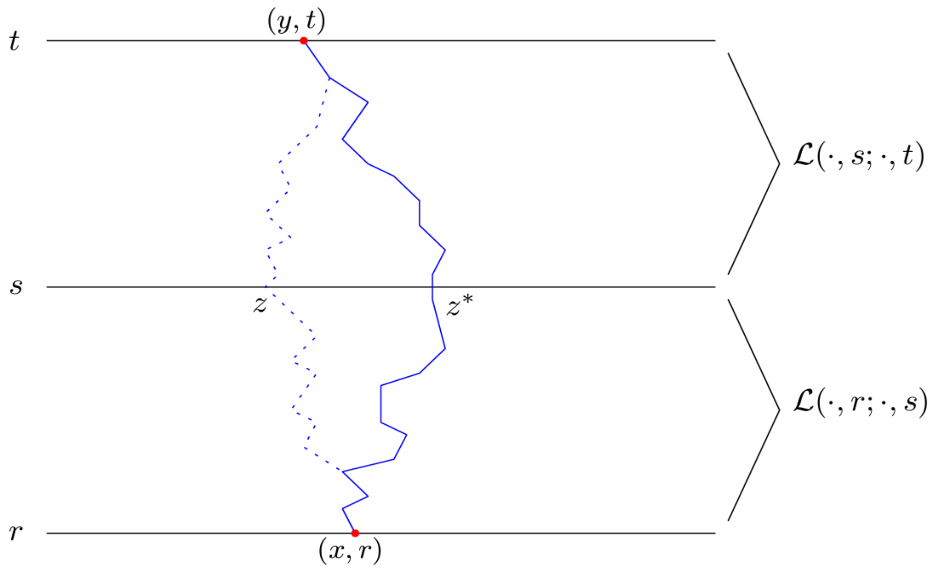
The limit a four-parameter random function

$\mathcal{L}\colon \mathbb{R}^4_\uparrow \to \mathbb{R}$, where $\mathbb{R}^4_\uparrow = \{(x, s; y, t) \in \mathbb{R}^4 : s < t\}$

(the **directed landscape**)

Composition law $\mathcal{L}(x, r; y, t) = \max_z \mathcal{L}(x, r; z, s) + \mathcal{L}(z, s; y, t)$

This gives a *directed metric* on $\mathbb{R}^2$



Random metric generates growth process

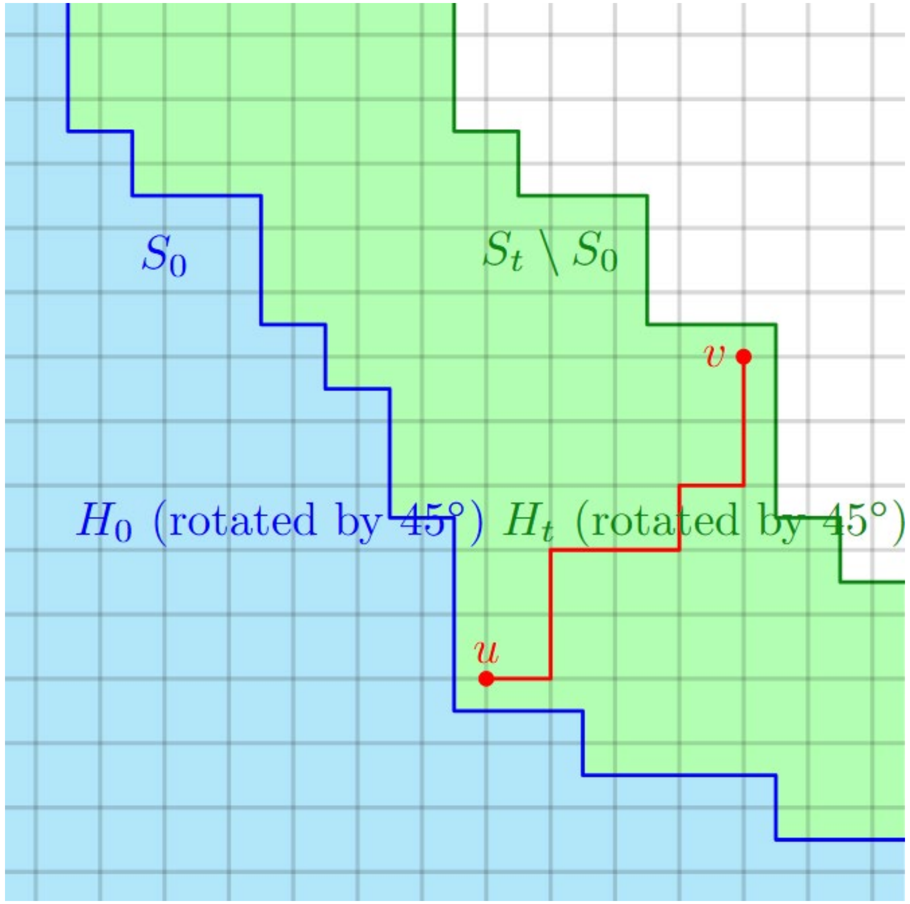
Growth process $(H_t)_{t\geq 0}$ starting from general H_0: take S_0 according to H_0; define

$$S_t = \{v \in \mathbb{Z}^2 : \max_{u \notin S_0} T_{u,v} < t\}$$

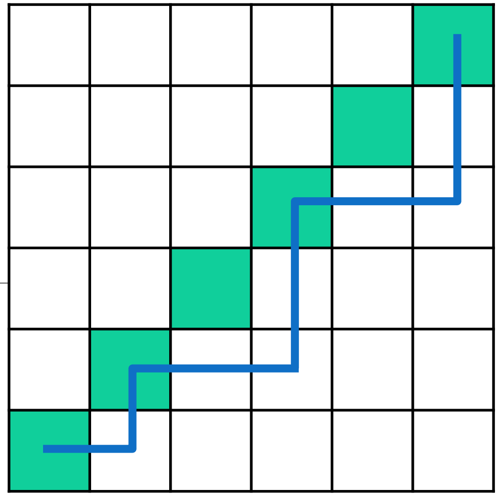
Therefore, we have **variational formula**:

$$\mathfrak{h}_t f(y) = \max_x f(x) + \mathcal{L}(x, 0; y, t)$$

i.e., directed landscape generates KPZ fixed point.



Scaling limit in half-space



Recall setup:

- $\xi(i,j) = \xi(j,i) \sim \mathrm{Exp}(1)$, for $i > j \in \mathbb{Z}$
- $\xi(i,i) \sim \mathrm{Exp}(a)$, for $i \in \mathbb{Z}$

(Baik-Barraquand-Corwin-Suidan, 16') For fixed a:

$$\mathbb{P}\left(T^a_{(1,1),(n,n)} < 4n + x2^{4/3}n^{1/3}\right) \to F_{GSE}(x), \qquad \text{if } a > 1/2,$$

$$\mathbb{P}\left(T^a_{(1,1),(n,n)} < 4n + x2^{4/3}n^{1/3}\right) \to F_{GOE}(x), \qquad \text{if } a = 1/2,$$

$$\mathbb{P}\left(T^a_{(1,1),(n,n)} < na^{-1}(1-a)^{-1} + x\sigma_a n^{1/2}\right) \to G(x), \qquad \text{if } a < 1/2, \quad \text{(Gaussian)}$$

Scale a around $1/2$: take $a = (1 - 2^{-1/3}\theta n^{-1/3})/2$, for some $\theta \in \mathbb{R}$:

$$\mathbb{P}\left(T^a_{(1,1),(n,n)} < 4n + x2^{4/3}n^{1/3}\right) \to F_{cross,\theta}(x),$$ formula given by Fredholm Pfaffian

Also have convergence of the process $2^{-4/3}n^{-1/3}\left(T^a_{(1,1),(n+x(2n)^{2/3},n-x(2n)^{2/3})}\right)$, $x \geq 0$.

Prediction goes back to *Kardar, 85'*. See also *Baik-Rains, 99'*, *Sasamoto-Imamura, 03'*, in different settings.

Half-space KPZ fixed point

Height function $H_t: \mathbb{Z}_{\geq 0} \to \mathbb{Z}$, with $H_t(x+1) = H_t(x) \pm 1$,

➢ Markov process : flip with rate 1 independently, except for rate a at 0.

For $\left(H_t^{(\varepsilon)}\right)_{t \geq 0}$, scaling: $h_t^\varepsilon(x) = -\varepsilon^{1/2}\left(H^{(\varepsilon)}_{2\varepsilon^{-3/2}t}(2\varepsilon^{-1}x) - \varepsilon^{-3/2}t\right)$

Take either

- $a = (1 - \theta\varepsilon^{1/2})/2$, or
- $a > 1/2$, fixed. (Let $\theta = -\infty$ in this case.)

(X. Zhang, 24') As $\varepsilon \to 0$, assume that h_0^ε converges to $f: \mathbb{R}_{\geq 0} \to \mathbb{R} \cup \{-\infty\}$. Then for $t > 0$, h_t^ε converges (in finite dim distribution) as $\varepsilon \to 0$, to $\mathfrak{h}_t^\theta f$, the **half-space KPZ fixed point** starting from f at time t.

Again, given by exact formulas (Fredholm Pfaffian).

Ganguly, S. Random metric geometries on the plane and Kardar-Parisi-Zhang universality. *AMS Notices (Jan 2022)*

Half-space directed landscape

(Dauvergne-Z., 26)

As $n \to \infty$

$(x, s; y, t) \mapsto 2^{-4/3} n^{-1/3} \left(T^a_{R_n(x,s),R_n(y,t)} - 4n(t-s) - 2^{8/3} n^{2/3}(y-x)\right)$ converges,

where $R_n(x, s) = (ns + 2^{5/3} n^{2/3} x, ns)$

The limit a four-parameter random function

$\mathcal{L}^\theta\colon \mathbb{R}^4_{+,\uparrow} \to \mathbb{R}$, where $\mathbb{R}^4_{+,\uparrow} = \{(x, s; y, t) \in \mathbb{R}^4 : x, y \geq 0;\ s < t\}$

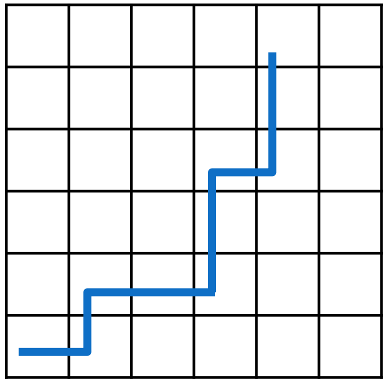
(the **half-space directed landscape with parameter** $\theta \in \mathbb{R} \cup \{-\infty\}$)

Composition law $\mathcal{L}^\theta(x, r; y, t) = \max_{z \geq 0} \mathcal{L}^\theta(x, r; z, s) + \mathcal{L}^\theta(z, s; y, t)$

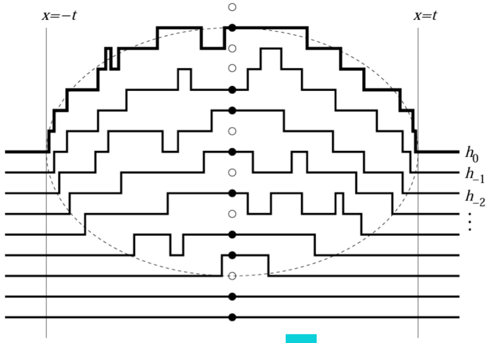
Variational formula $\mathfrak{h}^\theta_t f(y) = \max_{x \geq 0} f(x) + \mathcal{L}^\theta(x, 0; y, t)$

Previous route to landscape

In full-space, consider line ensembles

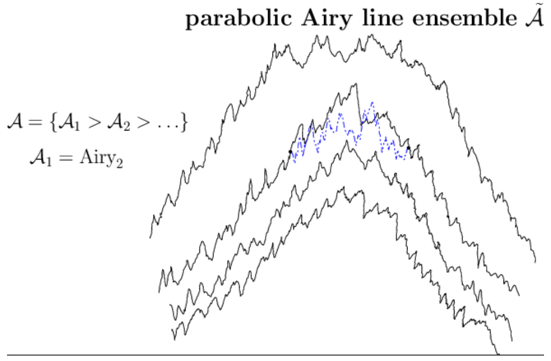


RSK correspondence

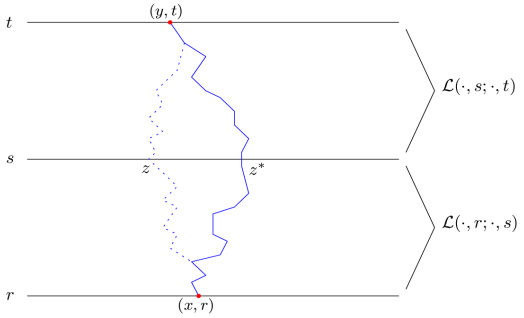


Some Gibbsian line ensemble (i.e., non-intersecting random walks)

convergence to



Prähofer-Spohn, 02'
Corwin-Hammond, 09';
characterization by
Aggarwal-Huang, 23'



Construct landscape *(Dauvergne-Ortmann-Virag, 18', Dauvergne-Virag, 21', Virag-Wu, 25')*

Concurrent works in progress to get half-space directed landscape via line ensembles.

Alternative route

Characterize directed landscape from KPZ fixed point:

(Dauvergne, Z., 24') For random function $\mathcal{M}: \{(x,s;y,t): s<t\} \to \mathbb{R}$ that is continuous, consider the following conditions:

1. For any $s<t$ and $f: \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$, supported on finitely many points, we have (in law)
$$\sup_x \mathcal{M}(x,s;\cdot,t) + f(x) = \mathfrak{h}_{t-s} f,$$
2. For any $r<s<t$, and x,y,z, we have $\mathcal{M}(x,r;y,t) \geq \mathcal{M}(x,r;z,s) + \mathcal{M}(z,s;y,t)$,
3. For any disjoint intervals $\{(s_i,t_i)\}_{i=1}^k$, $\mathcal{M}(\cdot, s_i; \cdot, t_i)$ are independent.
4. For the stationary horizon $\{\mathcal{H}_i\}_{i=1}^k$, we have $\left\{\sup_x \mathcal{M}(x,s;\cdot,t) + \mathcal{H}_i(x) - C_i\right\}_{i=1}^k = \{\mathcal{H}_i\}_{i=1}^k$ in law.

If $\mathcal{M}$ satisfies 1, 2, 3, and $\mathcal{M}'$ satisfies 1, 2, 3, 4, then they are equal in law.

Corollary Exponential LPP converges under 1:2:3 scaling.

Proof Tightness follows from KPZ fixed point convergence; any subsequential limit satisfies 1, 2, 3, 4.

Corollary Any $\mathcal{M}$ satisfies 1, 2, 3 must has the same law as the directed landscape $\mathcal{L}$.

Stationary horizon (full-space)

Brownian motion is *stationary* to the directed landscape:
for $B: \mathbb{R} \to \mathbb{R}$ being a two-sided Brownian motion with slope λ,
$\sup_x \mathcal{L}(x, 0; \cdot, t) + B(x) - C$ is a Brownian motion with slope λ, where $C = \sup_x \mathcal{L}(x, 0; 0, t) + B(x)$.

- How about multiple slopes?

Given any $\lambda_1 > \cdots > \lambda_k$, there should be a family of processes $\{\mathcal{H}_i\}_{i=1}^k$, where each $\mathcal{H}_i$ is a two-sided Brownian motion with slope λ_i. Also, they are jointly stationary:

$$\left\{\sup_x \mathcal{L}(x, s; \cdot, t) + \mathcal{H}_i(x) - C_i\right\}_{i=1}^k = \{\mathcal{H}_i\}_{i=1}^k$$

Busani, Seppäläinen, Sorensen, 2021-2022

Such $\{\mathcal{H}_i\}_{i=1}^k$ can be defined as follows:

- Consider two sided Brownian motions $B_1, \ldots, B_k$, with drifts $\lambda_1 > \cdots > \lambda_k$.
- Let $\mathcal{H}_i(x) - \mathcal{H}_i(y) = \lim_{z \to -\infty} B[(z, i) \to (x, k)] - B[(z, i) \to (y, k)]$

Here $B[u \to v]$ is the Brownian LPP passage time across $B_1, \ldots, B_k$.

Half-space characterization

(Dauvergne, Z., 26) Fix $\theta \in \mathbb{R} \cup \{-\infty\}$. For random function $\mathcal{M}: \{(x, s; y, t): x, y \geq 0; s < t\} \to \mathbb{R}$ that is continuous, consider the following conditions:

1. For any $s < t$ and $f: \mathbb{R}_{\geq 0} \to \mathbb{R} \cup \{-\infty\}$, supported on finitely many points, we have (in law)

$$\sup_{x \geq 0} \mathcal{M}(x, s; \cdot, t) + f(x) = \mathfrak{h}^{\theta}_{t-s} f,$$

2. For any $r < s < t$, and $x, y, z \geq 0$, we have $\mathcal{M}(x, r; y, t) \geq \mathcal{M}(x, r; z, s) + \mathcal{M}(z, s; y, t)$,
3. For any disjoint intervals $\{(s_i, t_i)\}_{i=1}^{k}$, $\mathcal{M}(\cdot, s_i; \cdot, t_i)$ are independent.
4. For the half-space stationary horizon $\{\mathcal{H}^{\theta}_i\}_{i=1}^{k}$, we have $\left\{\sup_{x \geq 0} \mathcal{M}(x, s; \cdot, t) + \mathcal{H}^{\theta}_i(x) - C_i\right\}_{i=1}^{k} = \{\mathcal{H}^{\theta}_i\}_{i=1}^{k}$ in law.

If $\mathcal{M}$ satisfies 1, 2, 3, and $\mathcal{M}'$ satisfies 1, 2, 3, 4, then they are equal in law.

Corollary Half-space Exponential LPP converges under 1:2:3 scaling. (Using half-space KPZ fixed point convergence.)

Corollary Any $\mathcal{M}$ satisfies 1, 2, 3 must has the same law as the half-space directed landscape $\mathcal{L}^{\theta}$.

Half-space stationary horizon

More complicated stationary processes (*Corwin-Barraquand, 18',* in LPP and polymer settings.)

- when $\theta \leq 0$, there is one with slope λ, for each $\lambda \in \mathbb{R}_{\geq 0}$;
- when $\theta > 0$, there is one with slope λ, for each $\lambda \in \{-2\theta\} \cup (2\theta, \infty)$;
- in either case, it is a Brownian motion when $\lambda = -2\theta$.

		$Exp(\lambda_1/2-\theta)$	$\widetilde{B_1}$
	…	…	…
$Exp(\lambda_k/2-\theta)$	…	$Exp(\lambda_1/2+\lambda_k/2)$	$\widetilde{B_k}$
0	…	$Exp(\lambda_1/2-\lambda_k/2)$	B_k
	…	…	…
		0	B_1

Dauvergne-Z., 25+

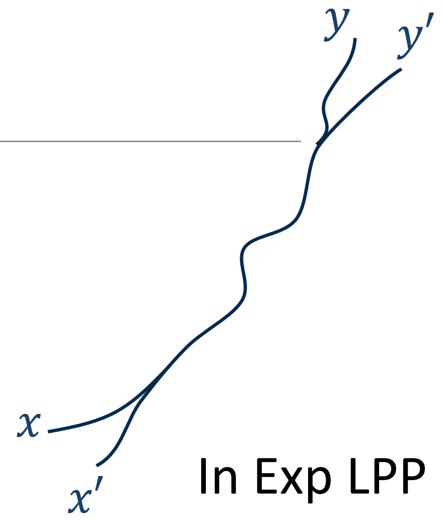
- Take Brownian motions $B_1, \dots, B_k, \widetilde{B_k}, \dots, \widetilde{B_1}$, with slopes $\lambda_1 > \cdots > \lambda_k \geq 0 \geq -\lambda_k > \cdots > -\lambda_1$. Also, $\lambda_k \geq \theta$.
- Let $\mathcal{H}_i^\theta(x) = TB[(k+1-i, i) \to (x, 2k)]$;

Here $TB[u \to v]$ denotes Exponential LPP followed by Brownian LPP

- Joint stationary

$$\left\{\sup_x \mathcal{L}^\theta(x, s; \cdot, t) + \mathcal{H}_i^\theta(x) - C_i\right\}_{i=1}^k = \{\mathcal{H}_i^\theta\}_{i=1}^k$$

Proof ideas & ingredients



Heuristics (of why such characterizations hold?)
Geodesics coalesce.

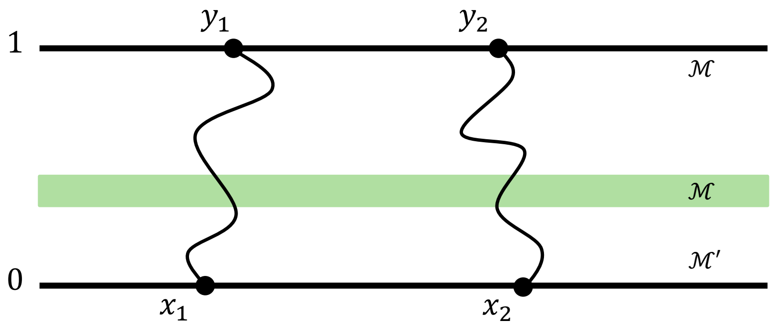
For $|x - x'|$ and $|y - y'|$ small, likely

$$\mathcal{M}(x,0;y,1) + \mathcal{M}(x',0;y',1) = \mathcal{M}(x,0;y',1) + \mathcal{M}(x',0;y,1)$$

Proof of characterization to show in law $\big(\mathcal{M}(x_1,0;y_1,1), \mathcal{M}(x_2,0;y_2,1)\big) = \big(\mathcal{M}'(x_1,0;y_1,1), \mathcal{M}'(x_2,0;y_2,1)\big)$

Use a *Lindeberg exchange strategy,* plus careful quantitative analysis, to exploit **coalescence**

(Originated in Lindeberg's proof of CLT, 1922; used in e.g., hydrodynamics *Bohadoran-Guiol-Ravishankar-Saada, 02'*; spin glass *Chatterjee, 04'*; random matrices: *Chatterjee, 05', Tao-Vu, 07', Knowles-Yin, 17'*; (2+1)D growth, *Caravenna-Sun-Zygouras, 21', Tsai, 24'*)



Equal with high probability
(under a certain coupling)