Perturbation of TASEP:
infinite order transition in the slow bond problem

joint with Allan Sly (Princeton) and Sourav Sarkar (Cambridge)

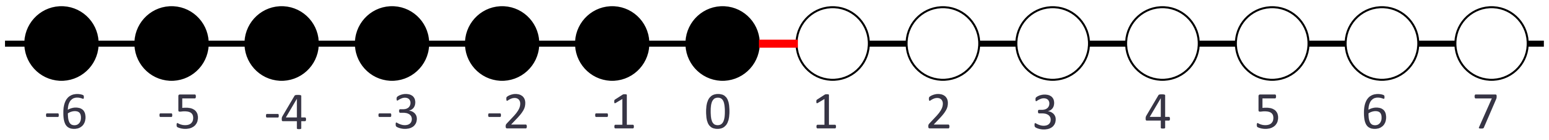
张灵夫 Lingfu Zhang

Department of Statistics and the Miller Institute, UC Berkeley

The second International Conference for Chinese Young Probability Scholars,
第二届华人青年概率学者国际会议 **Apr 30, 2023**

The model of TASEP

Totally Asymmetric Simple Exclusion Process (TASEP)

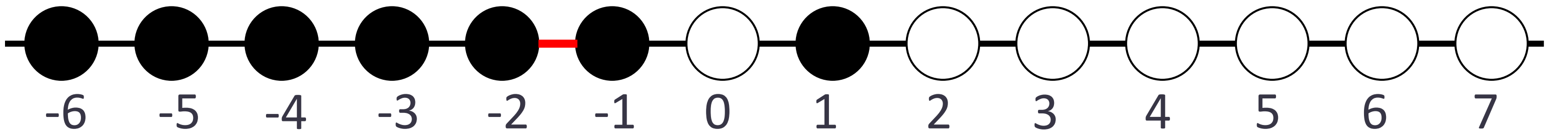


Step initial: particles at non-positive sites

Each edge rings according to a Poisson clock with rate 1

The model of TASEP

Totally Asymmetric Simple Exclusion Process (TASEP)

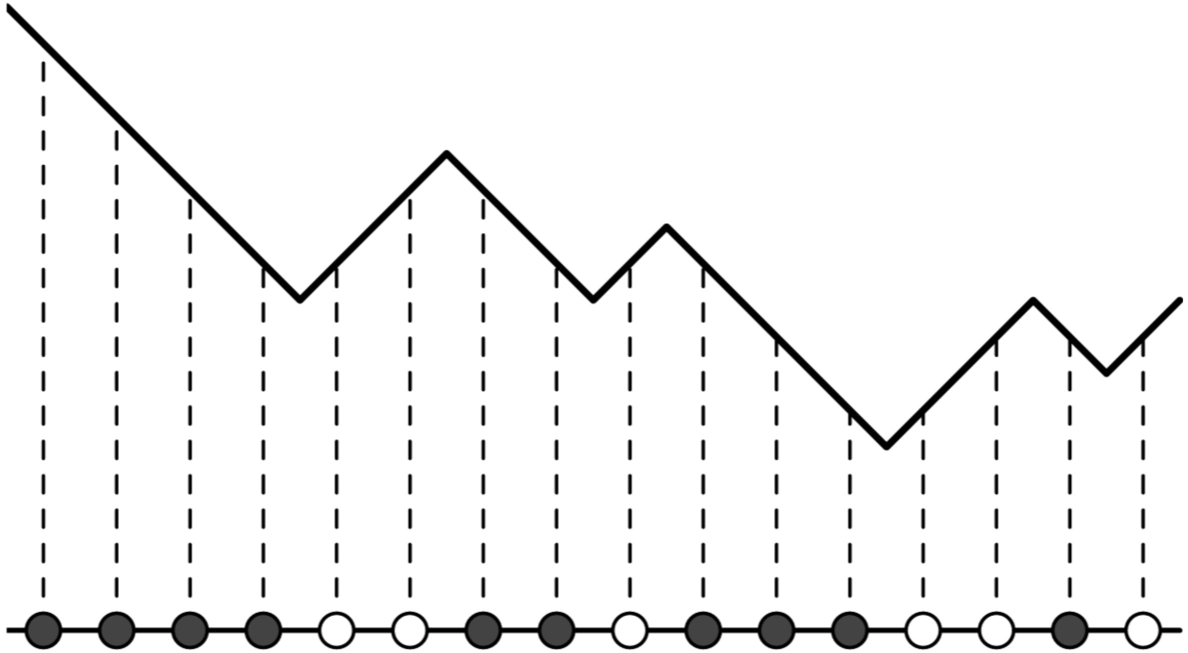


Step initial: particles at non-positive sites

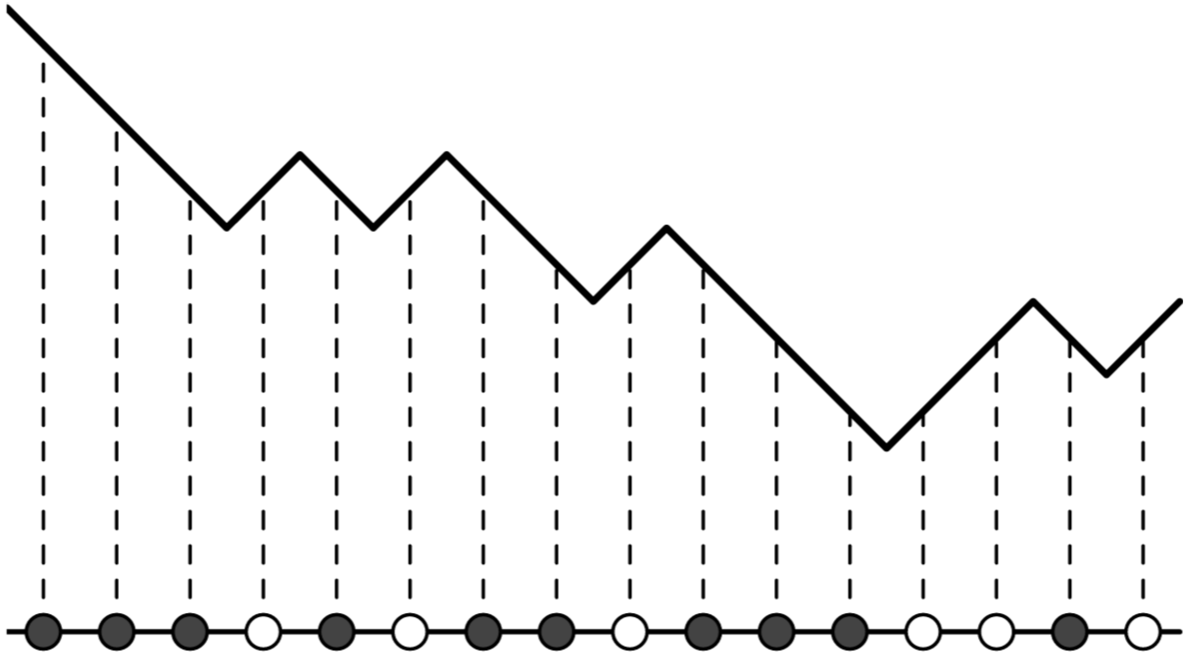
Each edge rings according to a Poisson clock with rate 1

Blocked if already occupied by another particle

Growth Surface representation



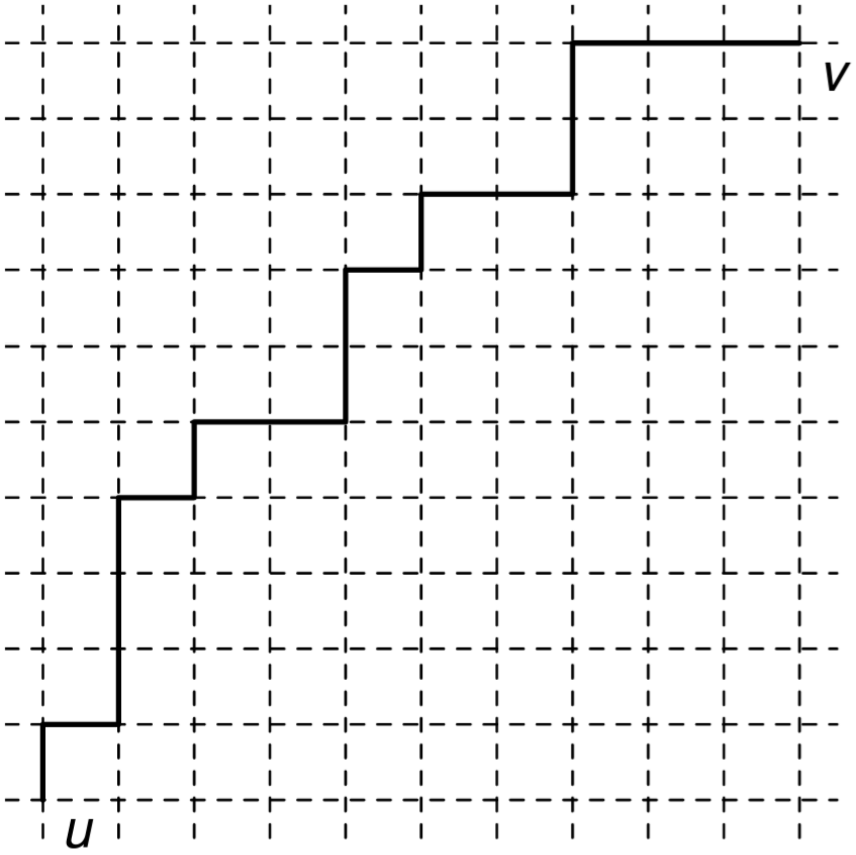
Growth Surface representation



Surface is in the Kardar-Parisi-Zhang (KPZ) universality class

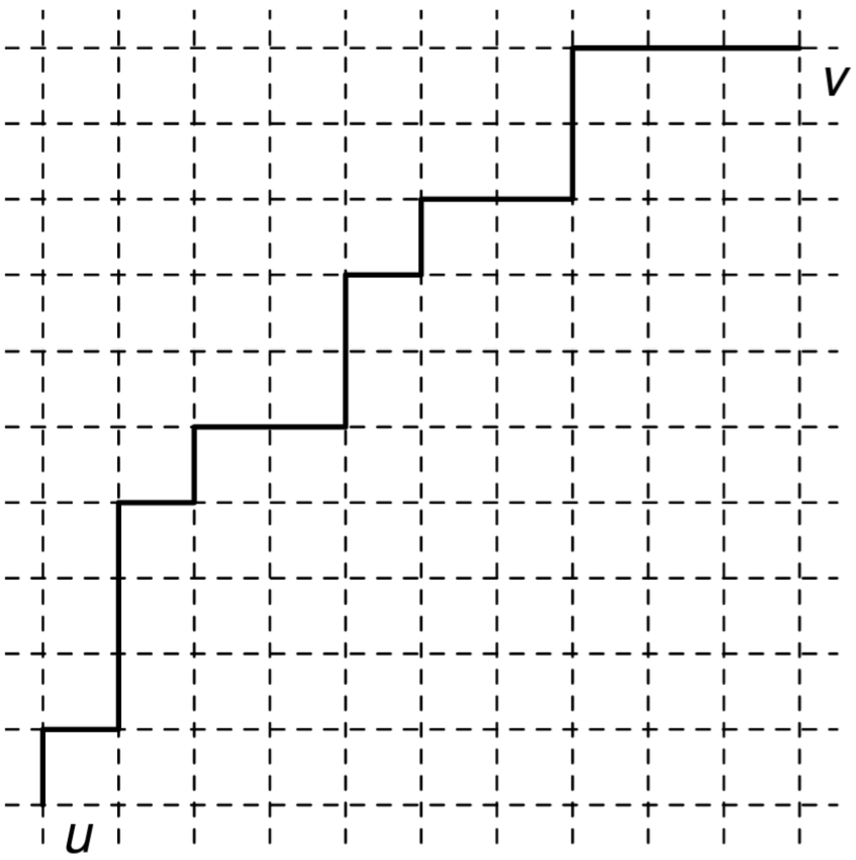
Step initial: growth from a corner

Last Passage Percolation



- $\xi(v) \sim \text{Exp}(1)$, i.i.d. $\forall v \in \mathbb{Z}^2$
- Passage time: $T_{u,v} := \max_\gamma \sum_{w\in\gamma} \xi(w)$
- Geodesic: $\Gamma_{u,v} := \text{argmax}_\gamma \sum_{w\in\gamma} \xi(w)$

Last Passage Percolation

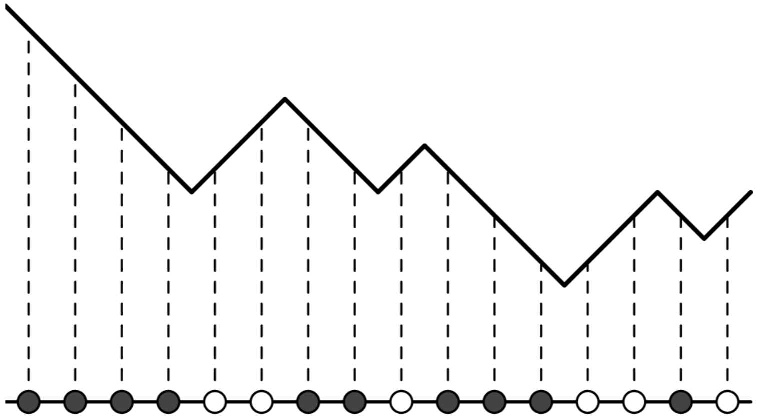


With growth surface:

$T_{(0,0),(i,j)}$

= time when surface reaches (i, j)

= time when the particle from $-i$ makes the $(j+1)$-th jump



Exactly solvable in the KPZ universality class.
Some classical results:

- $T_{(0,0),(n,n)} \sim 4n$ (Rost, 1981).

$T_{(0,0),(n,n)}$: time when the particle from $-n$ jumps out of 0

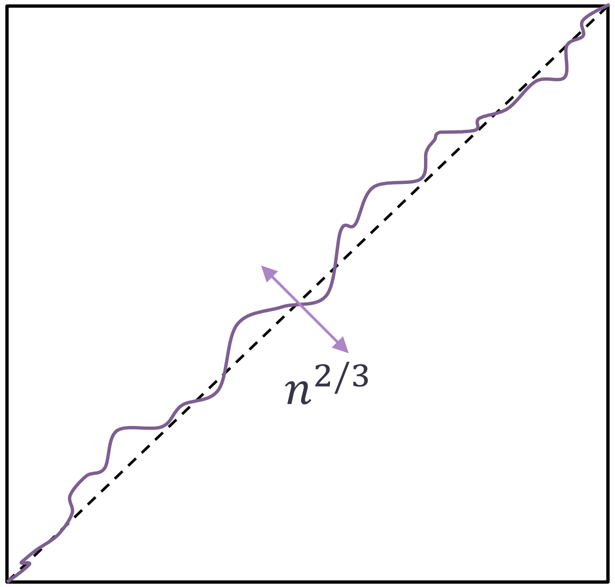
- Max stationary current of $\frac{1}{4}$

Exactly solvable in the KPZ universality class.
Some classical results:

- $T_{(0,0),(n,n)} \sim 4n$ (Rost, 1981).
- $2^{-4/3}n^{-1/3}(T_{(0,0),(n,n)} - 4n)$ converges weakly to the GUE Tracy-Widom distribution (Johansson, 2000).

$T_{(0,0),(n,n)}$: time when the particle from $-n$ jumps out of 0

➢Max stationary current of $\frac{1}{4}$



Geometry:
(Johansson, 2000) Transversal fluctuation is of order $n^{2/3}$

Exactly solvable in the KPZ universality class.
Some classical results:

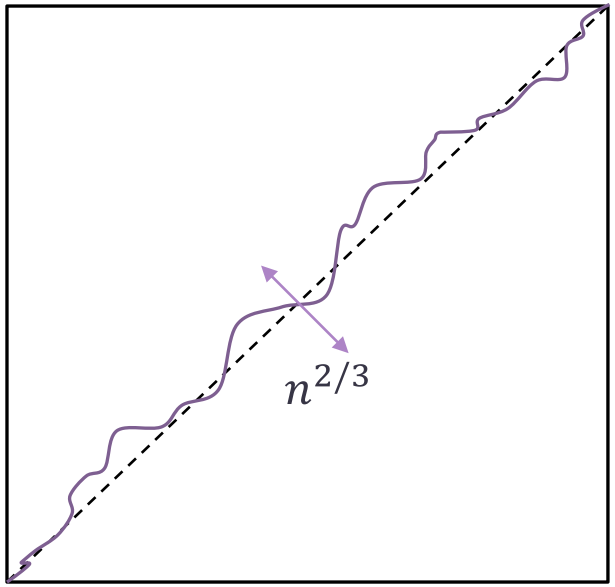
- $T_{(0,0),(n,n)} \sim 4n$ (Rost, 1981).
- $2^{-4/3}n^{-1/3}(T_{(0,0),(n,n)} - 4n)$ converges weakly to the GUE Tracy-Widom distribution (Johansson, 2000).
- Point to line profile: stationary Airy$_2$ process minus a parabola (Borodin and Ferrari, 2008)

$$2^{-4/3}n^{-1/3}\left(T_{(0,0),(n-x(2n)^{2/3},n+x(2n)^{2/3})} - 4n\right) \Rightarrow \mathcal{A}_2(x) - x^2$$

  $\mathcal{A}_2$ is absolute continuous with respect to Brownian motion (Corwin and Hammond, 2014).
- General initial data: KPZ fixed point (Matetski, Quastel, and Remenik, 2017).
  Joint scaling limit: the directed landscape (Dauvergne, Ortmann, and Virág, 2018).

$T_{(0,0),(n,n)}$: time when the particle from $-n$ jumps out of 0

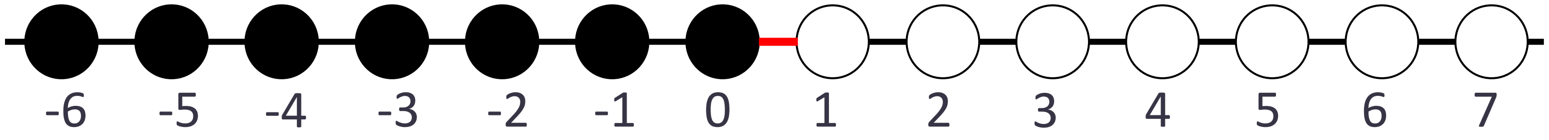
- Max stationary current of $\frac{1}{4}$



Geometry:
(Johansson, 2000) Transversal fluctuation is of order $n^{2/3}$

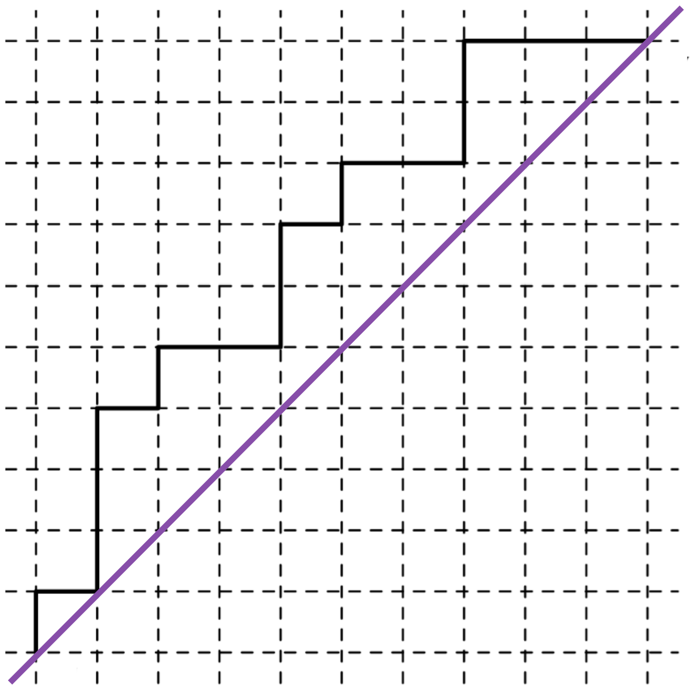
Local defects

Single slow bond in TASEP



Poisson clocks:

- rate $1-\epsilon$ for the edge between 0 and 1;
- rate 1 elsewhere



In LPP:

$\mathrm{Exp}(1-\epsilon)$ along diagonal, $\mathrm{Exp}(1)$ elsewhere

Note $\mathrm{Exp}(1-\epsilon) = \mathrm{Exp}(1) + \mathrm{Bernoulli}(\epsilon)\mathrm{Exp}(1-\epsilon)$

Single slow bond in TASEP

$L_n^\epsilon = T_{(0,0),(n,n)}^\epsilon$: time when the particle from $-n$ moves out of 0

- Then $\mathrm{C}(\epsilon) = \lim_{n\to\infty} \frac{n}{L_n^\epsilon}$ is the maximum stationary current (across 0)

Question (back to Janowsky and Lebowitz, 1992):

is there a macroscopic slow down?

i.e., is $\mathrm{C}(\epsilon) < \frac{1}{4}$ for any $\epsilon > 0$?

Two reasons for it being difficult:

1) Hard to simulate
2) Perturbation destroies exact-solvable structures

Simulations and predictions

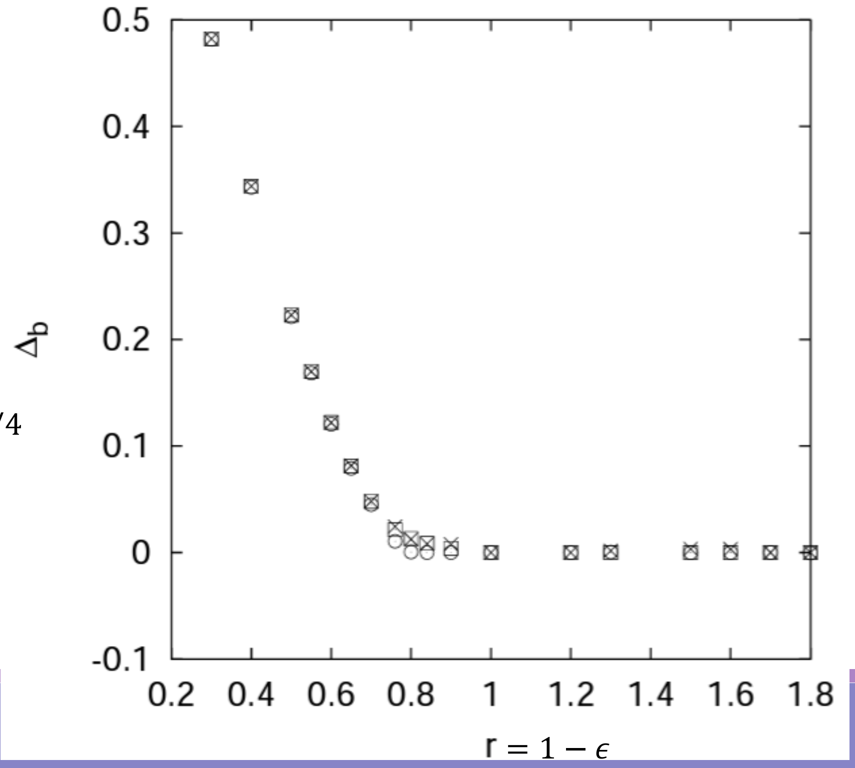
Disagreement among physicists

Janowsky and Lebowitz, 1994: $C(\epsilon) < \frac{1}{4}$ for any $\epsilon > 0$, by heuristics

Ha, Timonem, den Nijs, 2003: simulations

$C(\epsilon) = \frac{1}{4}$ for $\epsilon < 0.2$ (roughly)

$C(\epsilon) = (1 - \Delta_b^2)/4$



Rigorous bounds

Janowsky and Lebowitz, 1994: $C(\epsilon) < \frac{1}{4}$ for any $\epsilon < 0.49$

Seppäläinen, 2001: $\frac{1-\epsilon}{4-\epsilon} \leq C(\epsilon) \leq \min\{\frac{1}{4}, \frac{2(1-\epsilon)(2-\epsilon)}{(1-\epsilon)^2+2(2-\epsilon)}\}$

A closely related problem:

Baik and Rains, 2001:

longest increasing subsequence with involution with fixed points

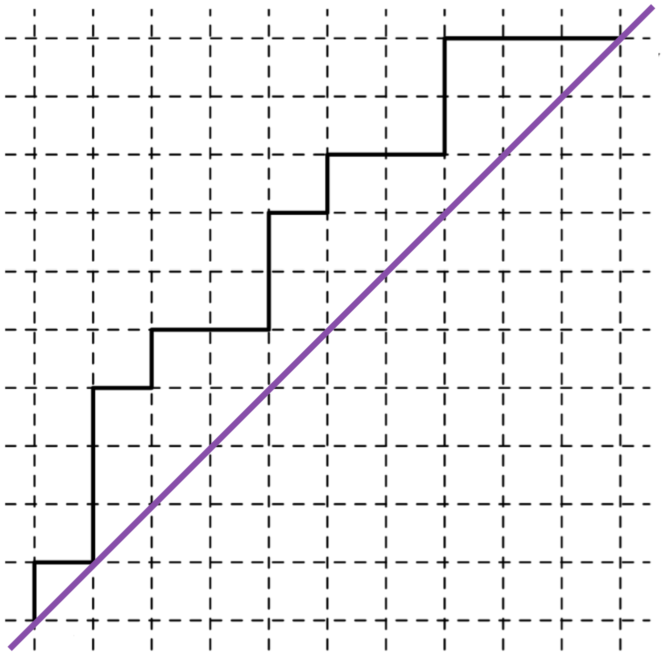
Non-trivial transition

Baik-Rains model

They consider random involution;

in the LPP setting, this means a symmetric environment

$$\xi(i,j) = \xi(j,i) \sim \text{Exp}(1)$$
$$\xi(i,i) \sim \text{Exp}(\lambda)$$



Then by symmetry, the optimal path can be taken in a half space

With other things, Baik and Rains, 2001 showed that

- when $\lambda \geq \frac{1}{2}$, current is still $\frac{1}{4}$
- when $\lambda < \frac{1}{2}$, current is $< \frac{1}{4}$

Using algebraic formulas

Slow bond problem results

Theorem (Basu, Sidoravicius, and Sly, 2014)

$\mathrm{C}(\epsilon) < \frac{1}{4}$ for any $\epsilon > 0$. And L_n^ϵ has $n^{1/2}$ times Gaussian fluctuation.

Geodesic now has $O(1)$ typical and $\log(n)$ maximum transversal fluctuation

- Janowsky and Lebowitz, 1994 made the correct prediction; numerical stimulations are inaccurate
- Why?

Theorem (Sarkar, Sly, and Z., 2021)

For any $k > 0$, $\frac{1}{4} - \mathrm{C}(\epsilon) < \epsilon^k$ for any $\epsilon > 0$ small enough.

Ideas for $C(\epsilon) < \frac{1}{4}$

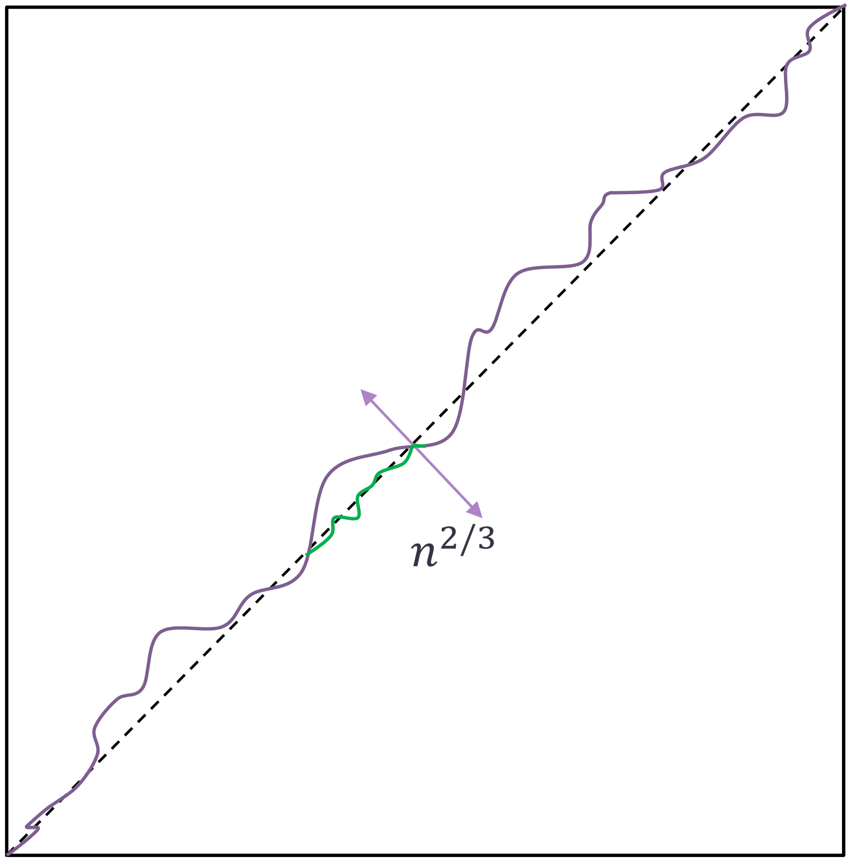
Superadditivity: $T_{(0,0),(m,m)} + T_{(m+1,m+1),(n,n)} < T_{(1,1),(n,n)}$

- Therefore, suffices to show that, for given $\epsilon > 0$, there exists n such that $\mathbb{E}L_n^{\epsilon} = \mathbb{E}T_{(0,0),(n,n)}^{\epsilon} > 4n$

Recall that $L_n = 4n + n^{1/3}X$, where $X \sim$ GUE Tracy-Widom, and it is known that $\mathbb{E}X < 0$

It then suffices to gain $Cn^{1/3}$ in expectation for some large C

Ideas for $c(\epsilon) < \frac{1}{4}$



Couple together using
$\text{Exp}(1-\epsilon) = \text{Exp}(1) + \text{Bernoulli}(\epsilon)\text{Exp}(1-\epsilon)$

Need to gain $Cn^{1/3}$ in expectation

As fluctuation is $n^{2/3}$, should spend $n^{1/3}$ time on diagonal. Then already gain $\sim \epsilon n^{1/3}$

Increase the time on diagonal by taking local deviations, in a multi-scale way

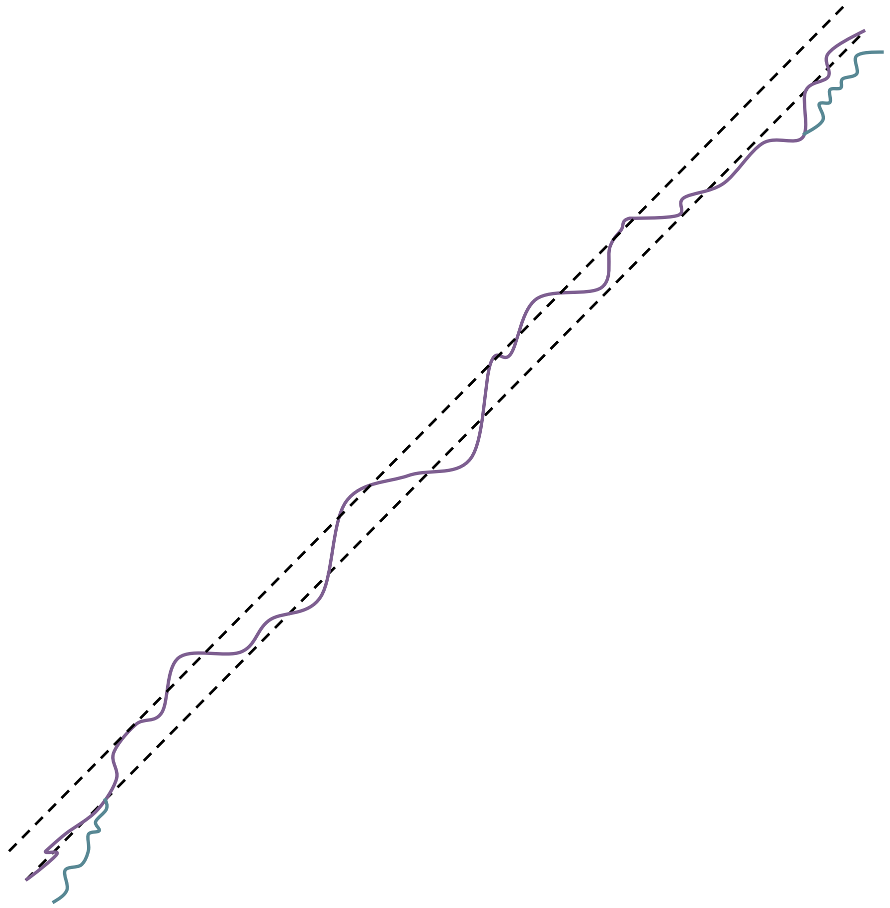
Bound $\frac{1}{4} - C(\epsilon)$

Main task: upper bound the time Γ_n^ϵ (the geodesic in the reinforced environment) spends in the diagonal

For the original geodesic, it spends $n^{1/3}$ time on diagonal:

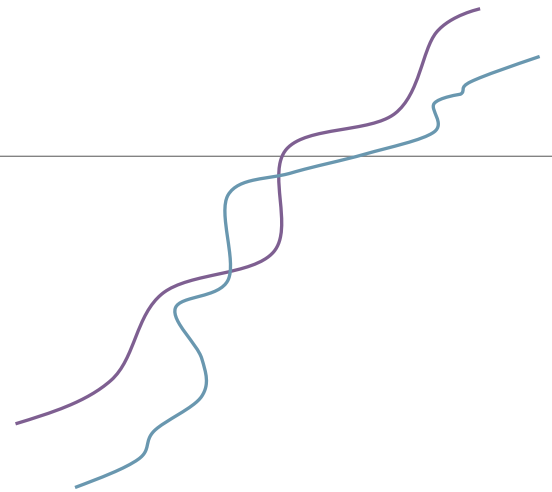
coalescence + translation invariant

❖ Then the geodesic spends roughly the same amount of time on each diagonal

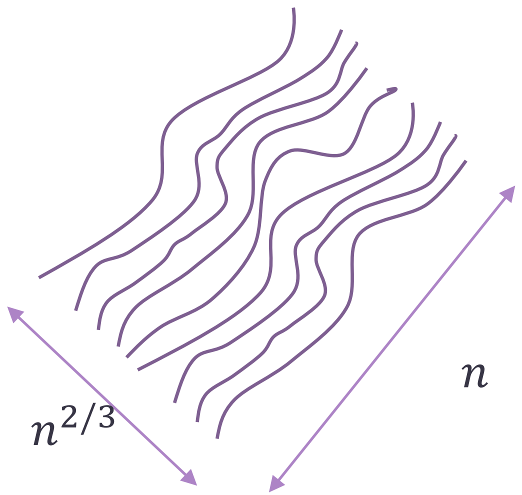


Why coalescence

Geodesics either coalescence
or stay disjoint; cannot be like:



An argument from Basu, Hoffman, Sly, 2018:
if geodesics (with nearby endpoints) are likely to stay disjoint, then will have:



and this can be ruled out, using negative expectation of GUE Tracy-Widom

Bound $\frac{1}{4} - C(\epsilon)$

Main task: upper bound the time Γ_n^ϵ (the geodesic in the reinforced environment) spends in the diagonal

For Γ_n^ϵ, if it is a ‘near’ geodesic (i.e., path with weights close to optimal), then can still bound its time on diagonal, as ‘highway picture still holds’

- ❖ Key difference 1: in proving coalescence, ‘near’ geodesics may cross each other twice
  - ➢ rank ‘near’ geodesics by total weights; same rank ones do not cross twice
- ❖ Key difference 2: multiple ‘near’ geodesics between same endpoints
  - ➢ Bound ‘multiple peak event’

Bound $\frac{1}{4} - \mathrm{C}(\epsilon)$

Two statements:

- If Γ_n^ϵ is a 'near' geodesic, then can bound its time on the diagonal
- If Γ_n^ϵ does not spend much time on the diagonal, then it is a 'near' geodesic

Induction in scales

For $\frac{1}{4} - \mathrm{C}(\epsilon) < \epsilon^k$, the number of scales depend on k.

A further question

Our arguments can be refined to obtain $\frac{1}{4} - \mathrm{C}(\epsilon) < \exp(-(\log(\epsilon^{-1}))^{a})$ for some $a > 1$

What is the actual order (as $\epsilon \to 0$)?

Costin, Lebowitz, Speer, Troiani, 2013 suggests $\frac{1}{4} - \mathrm{C}(\epsilon) \sim \exp(-c\epsilon^{-1})$

Thank you!