# Exclusion process

Saturday, May 10, 2025 2:54 PM

**Def** For countable $V$, $\eta: V \to \{0,1\}$ evolves such that for any $x \neq y$, $\eta(x)=1,\ \eta(y)=0 \Rightarrow \eta(y)=1,\ \eta(x)=0$ rate $P(x,y)$

Here $P(x,y) \geq 0$, $\sum_y P(x,y) = 1$, $\sum_x p(x,y) < \infty$

Interpretation: $\begin{cases} \eta(x)=1 & \text{particle at } x \\ \eta(x)=0 & x \text{ empty} \end{cases}$ exclusion: at most one particle at a location.

Question: Stationary? Well understood in symmetric case $(p(x,y)=p(y,x))$; less comprehensive in general settings.

1D lattice, nearest neighbor: exact-solvable; connections to random matrices

Other topics in 1D: queueing, multi-species, grand coupling.

- Stationary measures.

Example: ① If $p$ is doubly stochastic; i.e. $\sum_x p(x,y) = \sum_y p(y,x) = 1$, then iid Bernoulli$(p)$ for any $p \in [0,1]$ is stationary.

② If $p$ is reversible w.r.t. $\pi: V \to \mathbb{R}_{\geq 0}$, i.e., $\pi(x)P(x,y) = \pi(y)P(y,x)$

then independent Bernoulli$\left(\frac{\pi(x)}{1+\pi(x)}\right)$ is stationary.

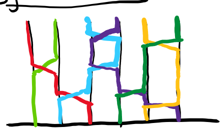
(proof: $\frac{\pi(x)}{1+\pi(x)} \cdot \frac{1}{1+\pi(y)} \cdot P(x,y) = \frac{\pi(y)}{1+\pi(y)} \cdot \frac{1}{1+\pi(x)} \cdot P(y,x)$ )

For $\mathbb{Z}$ with $p(x,x+1) = q$, $p(x+1,x) = 1-q$ (simple exclusion process on $\mathbb{Z}$)

can take $\pi(x) = c\left(\frac{q}{1-q}\right)^x$, and get a family of stationary.

[$q=\frac{1}{2} \Rightarrow$ reduces to ①; $q=1$: $\mathbb{1}[x \geq a]$ ]

Symmetric case. $p(x,y)=p(y,x)$; self-dual



Think of as a swap process: $\mathbb{P}[\eta_t = 1 \text{ on } A_0] = \mathbb{P}[\eta_0 = 1 \text{ on } A_t]$

For all extremal stationary: consider harmonic functions $h: V \to [0,1]$, $\sum_y p(x,y)h(y) = h(x)$, $x \in V$.

Theorem $\mu_h = \lim_{t\to\infty}$ starting from independent Bernoulli $(h(x))$

this limit exists; and $\{\mu_h\}_{\text{harmonic } h}$ gives all the extremal stationary measures.

proof of convergence

Take any $x_1, \cdots x_k$, and consider swap process with them as initial

$H(t) = \mathbb{E}[h(X_1(t)) \cdots h(X_k(t))] = \mathbb{P}[\eta_t(x_1) = \cdots \eta_t(x_k) = 1]$ via duality.

Moreover, $H(t)$ is non-increasing: $\begin{cases} k=1, & H(t) \text{ is constant} \\ k \geq 2, & \frac{d}{dt}H(t) = -\mathbb{E}\left[h(x_1(t)) \cdots h(x_k(t)) \sum_{i<j} \left(\frac{h(x_j(t))}{h(x_i(t))} - 1 + \frac{h(x_i(t))}{h(x_j(t))} - 1\right) P(x_j(t), x_i(t))\right] \leq 0 \end{cases}$

$\Rightarrow H(t)$ converges as $t \to \infty$

- Stationarity of $\mu_h$: obvious from convergence.
- Extremality? two cases:

Case 1. For independent walks (following $P$) starting from any $x, y$, almost surely $X(t) = Y(t)$ for some $t>0$, $i \neq j$.

Case 2. The remaining case. i.e. $\exists$ some walks with positive prob. no collide.

(case 1 implies recurrence of random walk)

proof in Case 1. Lemma. $\mathbb{P}[\eta(x_1) = \cdots = \eta(x_k) = 1]$ depends only on $k$.

Sketch proof Consider swap process starting from $x_1, \cdots, x_{k-1}, x_k$ and from $x_1, \cdots, x_{k-1}, x_k'$; they will "coalesce" after finite time, by condition above.

$\Rightarrow \mathbb{P}[\eta(x_1) = \cdots = \eta(x_k) = 1] = \mathbb{P}[\eta(x_1') = \cdots = \eta(x_k') = 1]$

by changing one-by-one

Therefore $\{\eta(x)\}_{x \in V}$ is an exchangeable distribution.

De Finetti's Theorem: any exchangeable distribution of $\{0,1\}^V$ is a mixture of Bernoulli$(p)$ iid.

Thus all extremal stationary all iid Bernoulli$(p)$

(Also in Case 1, all bounded harmonic functions must be constant, by considering two coalescing random walks)

proof of De Finetti's Thm.

For an exchangeable sequence $X_1, X_2, \cdots$, let $S_n = \frac{1}{n}\sum_{i=1}^n X_i$; then $\mathbb{E}S_n^k$ converges as $n \to \infty$, for any $k \in \mathbb{N}$

$\Rightarrow S_n$ converges in distribution (to some measure $\mu$, supp on $[0,1]$)

Consider the measure $\int \text{Bernoulli}(p)^{\mathbb{N}} d\mu(p)$; can check that $X_1, X_2, \cdots$ has the same moment as it, then given by this law.

proof in Case 2. Key Lemma: For harmonic $h$, and measure $\mu$, exclusion starting from $\mu$ converges to $\mu_h$ iff

$\sum_y P_t(x,y)\mu(\eta(y)=1) \to h(x) \quad \forall x$

$\sum P_t(x,y) P_t(x,z) \mu(\eta(y)=\eta(z)=1) \to h(x)^2 \quad \forall x \quad (\mathbb{P}[\eta_t(x)=1 \mid \eta_0 \sim \mu] \to h(x) \text{ in prob})$

(Idea: duality + coupling)

[Similar to Voter model; mixing in time]

Then for $\mu_h = \alpha\mu' + (1-\alpha)\mu''$, $\mu' \to \mu_h$ and $\mu'' \to \mu_h$; if stationary. $\mu' = \mu'' = \mu_h \Rightarrow \mu_h$ extremal

On the other hand, for any $\mu$ extremal, let $h(x) = \mathbb{E}_\mu[\eta(x)]$; want to show $\mu = \mu_h$.

Lemma. For $\mu$ extremal, $\mathbb{P}_\mu[\eta(x) = \eta(y) = 1] \leq \mathbb{P}_\mu[\eta(x)=1]\,\mathbb{P}_\mu[\eta(y)=1]$, $\forall x \neq y$.

This + non-collide (assumption of case 2) $\Rightarrow$ condition of Key Lemma

$\hookrightarrow$ proof Let $\mu_1$ be $\mu$ conditional on $\eta(x)=1$

$\mu_0$ — — — — $\eta(x)=0$

$\Rightarrow \mu = h(x)\mu_1 + (1-h(x))\mu_0$

Also $\eta_t | \eta_0 \sim \mu_0$, $\eta_t | \eta_0 \sim \mu_1 \to \mu$, by extremality of $\mu$

$\Rightarrow \sum_z P_t(y,z)\mu(\eta(x) = \eta(z) = 1) \to h(x)h(y)$, as $t \to \infty$

Thus if one starts independent walks from $x, y$ (denoted by $X(t), Y(t)$)

$\mathbb{P}_\mu[\eta(X(t)) = \eta(Y(t)) = 1] \to h(x)h(y)$

on the other hand, if one runs swap process from $x, y$ (denoted by $\tilde{X}(t), \tilde{Y}(t)$)

as shown above: $\mathbb{P}_\mu[\eta(\tilde{X}(t)) = \eta(\tilde{Y}(t)) = 1] \leq \mathbb{P}_\mu[\eta(X(t)) = \eta(Y(t)) = 1]$

by stationarity, $\mathbb{P}_\mu[\eta(x) = \eta(y) = 1] \leq h(x)h(y)$.

So far, all on symmetric case: self-duality crucially used.

General case: on $\mathbb{Z}^d$, assume translation invariant (i.e. $p(x,y) = p(0,y-x)$), and irreducible

(Then doubly stochastic since $\sum_x p(x,y) = \sum_x p(0,y-x) = 1$; iid Bernoulli$(p)$ are stationary)

Thm All translation invariant & stationary measures are iid Bernoulli$(p)$, or their mixture.

(Excludes Bernoulli$\left(\frac{\pi(x)}{1+\pi(x)}\right)$)

proof idea Take $\mu_p$ (iid Bernoulli$(p)$) and any other trans.inv. stationary $\mu$.

Can couple them together; e.g. start with $\mu_p$ and $\mu$ independently; run exclusion with the same Poisson clocks;

$\Rightarrow$ converge to coupled $\eta, \zeta$, s.t. $\eta \sim \mu_p$, $\zeta \sim \mu$. and $\mathbb{P}[\eta(x) = \zeta(y) = 1,\ \eta(y) = \zeta(x) = 0] = 0$.

(Because one can consider $\mathbb{P}[\eta(0) \neq \zeta(0)]$; otherwise this "differ prob" will further decrease)

Therefore $\mu_p$ either dominates $\mu$, or dominated by $\mu$, almost surely; if consider $\{\mu_p\}_{p\in[0,1]}$ all coupled together,

necessarily $\zeta \sim \mu$ equals $\eta \sim \mu_p$ for some $p$.