Free probability in RMT

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Back to global law, Question: sum/product of independent random matrices?

X Wigner, Y Wigner, $X+Y$ Wigner semi-circle + semi-circle $\Rightarrow$ semi-circle

$S=\frac{1}{n}XX^*$, $T=\frac{1}{n}YY^*$. $S+T=\frac{1}{n}[X|Y][X|Y]^*$ $MP\,\lambda$ + $MP\,\lambda$ $\Rightarrow$ $MP\,2\lambda$

(All these should be seen from moments already.)

More generally: semi-circle + MP? semi-circle + given measure? ...

The operation: free convolution $\mu \boxplus \nu$

Motivation: consider an $n\times n$ matrix X, normalized s.t. eigenvalues are of order 1.

Let $s=\frac{1}{n}\operatorname{tr}\frac{1}{X-zI_n}$ (Stieltjes transform); $\frac{1}{X-zI_n}=s(I_n-E)$, for $n\times n$ matrix E with $\operatorname{tr}E=0$

$\Rightarrow X = zI_n + \frac{1}{s(I_n-E)}$

For another $n\times n$ matrix Y, can find w, F, s.t. $Y=wI_n+\frac{1}{s(I_n-F)}$ (w, F determined by s)

i.e. $X+Y=(z+w)I_n+\frac{1}{s(I_n-E)}+\frac{1}{s(I_n-F)}=(z+w+s^{-1})I_n+s^{-1}(I_n-E)^{-1}(I_n-EF)(I_n-F)^{-1}$

$$\Rightarrow \frac{1}{X+Y-(z+w+s^{-1})I_n}=s(I_n-F)(I_n-EF)^{-1}(I_n-E)=s(I_n-F)(I_n+EF+EFEF+\dots)(I_n-E)$$

Take trace, assume that the RHS $\approx ns$ (can check $\frac{1}{n}\operatorname{tr}EF$, $\frac{1}{n}\operatorname{tr}FE$, $\frac{1}{n}\operatorname{tr}EFE$, ... all $\to 0$, via moments, for e.g. Wigner)

$\Rightarrow \frac{1}{n}\operatorname{tr}\frac{1}{X+Y-(z+w+s^{-1})I_n}\approx s$

i.e. given Stieltjes transform of X, Y, can compute Stieltjes transform of $X+Y$

$R_X: -s\mapsto z+s^{-1}$, $R_Y: -s\mapsto w+s^{-1}$

$\Rightarrow R_{X+Y}: -s\mapsto z+w+2s^{-1}$, $R_{X+Y}=R_X+R_Y$

More generally, for any prob measures μ, ν on $\mathbb{R}$

Stieltjes transform $T_\mu(z)=\int\frac{d\mu(x)}{x-z}$, $T_\nu(z)=\int\frac{d\nu(x)}{x-z}$

R-transform $R_\mu(-s)=T_\mu^{-1}(s)+s^{-1}$, $R_\nu(-s)=T_\nu^{-1}(s)+s^{-1}=\sum_{i=0}^{\infty}k_{i+1}(-s)^i$ $k_1, k_2, \dots$ free cumulants, determined by moments (e.g. $k_1=\mathbb{E}_\mu X$)

$\mu\boxplus\nu$ is the prob measure s.t. $R_{\mu\boxplus\nu}=R_\mu+R_\nu$ (free addition, or free convolution)

- For (renormalized) matrices $X^{(1)}, X^{(2)}, X^{(3)}, \dots$ suppose that $\frac{1}{n}\sum_{i=1}^n\delta_{\lambda_i(X^{(n)})}\to\mu$, and that they are "asymptotically free"
  $Y^{(1)}, Y^{(2)}, Y^{(3)}, \dots$ $\frac{1}{n}\sum_{i=1}^n\delta_{\lambda_i(Y^{(n)})}\to\nu$

  i.e. $\frac{1}{n}\operatorname{tr}\left[(P_1(X^{(n)})-\frac{1}{n}\operatorname{tr}P_1(X^{(n)})I_n)(P_2(Y^{(n)})-\frac{1}{n}\operatorname{tr}P_2(Y^{(n)})I_n)\dots\right]\to 0$

  for any k polynomials $P_1, P_2, \dots, P_k$, (alternate applied to $X^{(n)}, Y^{(n)}, X^{(n)}, \dots$)

  Then $\frac{1}{n}\sum_{i=1}^n\delta_{\lambda_i(X^{(n)}+Y^{(n)})}\to\mu\boxplus\nu$.

An explanation of semi-circle & mp law.

- semi-circle is the "free Gaussian": for radius r semi-circle law, $T_{\mu_{sc,r}}(z)=-\frac{2}{r^2}(z-\sqrt{z^2-r^2})$; $T^{-1}_{\mu_{sc,r}}(s)=-\frac{1}{s}-\frac{sr^2}{4}$; $R_{\mu_{sc,r}}(-s)=-\frac{sr^2}{4}$

For μ with $\underbrace{R_\mu(0)=0}_{(\mathbb{E}_\mu X=0)}$, $\underbrace{\frac{\mu}{\sqrt{N}}\boxplus\frac{\mu}{\sqrt{N}}\boxplus\dots\boxplus\frac{\mu}{\sqrt{N}}}_{N\text{ folds}}\xrightarrow[N\to\infty]{}$ semi-circle, since $R_{\frac{\mu}{\sqrt{N}}\boxplus\dots\boxplus\frac{\mu}{\sqrt{N}}}(s)=N R_{\frac{\mu}{\sqrt{N}}}(s)=\sqrt{N}R_\mu(s/\sqrt{N})\to c\cdot s$

- MP law is the "free Poisson":

  $[(1-\frac{\lambda}{N})\delta_0+\frac{\lambda}{N}\delta_\alpha]\boxplus\dots\boxplus[(1-\frac{\lambda}{N})\delta_0+\frac{\lambda}{N}\delta_\alpha]\to MP\,\lambda$, since $R_{\dots\boxplus\dots\boxplus\dots}(s)=N R_{(1-\frac{\lambda}{N})\delta_0+\frac{\lambda}{N}\delta_\alpha}(s)=N(\frac{\lambda}{N}(\frac{\alpha}{1-s\alpha})+O(\frac{1}{N^2}))\to\frac{\lambda\alpha}{1-s\alpha}$

  Solve Stieltjes transform, get $MP\,\lambda$, rescaled by α.

Similarly: $[(1-\frac{\lambda}{N})\delta_0+\frac{\lambda}{N}\mu]\boxplus\dots\boxplus[(1-\frac{\lambda}{N})\delta_0+\frac{\lambda}{N}\mu]\to$ compound $MP\,\mu,\lambda$ $R(s)=\int\frac{\lambda x}{1-sx}d\mu(x)$

(corresponds to sample cov. matrix, with general cov. struc; μ is covariance matrix spectral dist.)

Product of Matrices

S-transform: $S_\mu(z)=\frac{1+z}{z}\Psi_\mu^{-1}(z)$, $\Psi_\mu(w)=\int_0^\infty\frac{xw}{1-xw}d\mu(x)$

$\mu\boxtimes\nu$ is the measure with $S_{\mu\boxtimes\nu}(z)=S_\mu(z)\cdot S_\nu(z)$

For $X^{(1)}, X^{(2)}, \dots$ with $\frac{1}{n}\sum_{i=1}^n\delta_{\lambda_i(X^{(n)})}\to\mu$
$Y^{(1)}, Y^{(2)}, \dots$ with $\frac{1}{n}\sum_{i=1}^n\delta_{\lambda_i(Y^{(n)})}\to\nu$
$\Rightarrow \frac{1}{n}\sum_{i=1}^n\delta_{\lambda_i(\sqrt{X^{(n)}}Y^{(n)}\sqrt{X^{(n)}})}\to\mu\boxtimes\nu$, if $X^{(n)}, Y^{(n)}$ "asymptotic free", $X^{(n)}$ pos. semi-def.
[e.g. if $X^{(n)}$ is invariant under conjugacy by any unitary/orthogonal matrix]

Application: for sample covariance matrix with general covariance structure $\Rightarrow \mu_{mp}\boxtimes\mu_{cov}$, since can be written as $\Sigma^{\frac{1}{2}}XX^*\Sigma^{\frac{1}{2}}$, where XX^* is standard sample cov. matrix.

μ_{cov} $\hookrightarrow$ eigenvalues of covariance matrix Σ

(= compound $MP_{\mu_{cov}}$)