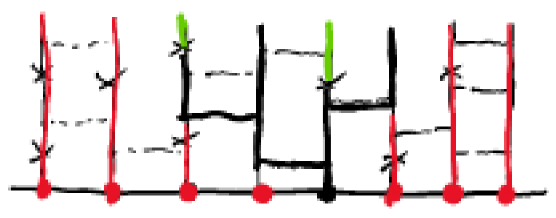
Infection models

Saturday, May 10, 2025 2:53 PM

For a graph $G=(V,E)$, each vertex has one of three possible status: susceptible S, infected I, removed R



- Each infected infects each susceptible neighbor at rate λ
- Each infected becomes removed with rate 1.

Question: is there going to be a outbreak? How long it lasts?
(A positive portion gets infected)

Simplest case: <u>complete graph</u> (well mixed population) interesting regime: infection rate λ/n

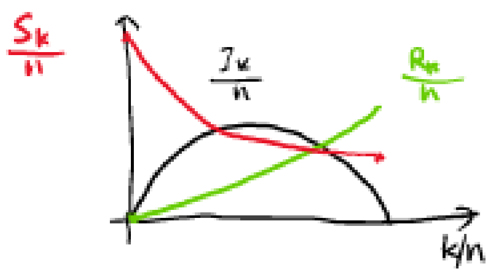
$S_k\ I_k\ R_k$: # after k-th change (change: one more infection, or one more recovery) $S_1=n-1$, $I_1=1$, $R_1=0$

$$\begin{cases} I_k+2R_k=k \\ S_k+I_k+R_k=n \end{cases} \Rightarrow R_k=\frac{k-I_k}{2},\quad S_k=n-\frac{k}{2}-\frac{I_k}{2}$$

$$I_{k+1}=\begin{cases} I_k+1 & \text{one more infection} & \text{rate } \frac{\lambda I_k S_k}{n} \\ I_k-1 & \text{one infected} \to \text{removed} & \text{rate } I_k \end{cases}$$

$\mathbb{P}[I_{k+1}=I_k+1\mid I_k]=\frac{\lambda S_k}{\lambda S_k+n}$; $\mathbb{E}[I_{k+1}-I_k\mid I_k]=\frac{\lambda S_k-n}{\lambda S_k+n}=\frac{\lambda(2n-k-I_k)-2n}{\lambda(2n-k-I_k)+2n}$. If $\lambda\le 1$, no outbreak w.h.p.

For $\lambda>1$, with pos. prob, would like



$f:\alpha\mapsto \frac{I_{\alpha n}}{n}$ solves $f'(\alpha)=\frac{\lambda(2-\alpha-f(\alpha))-2}{\lambda(2-\alpha-f(\alpha))+2}$

order n changes

time: $\sim\sum_k \frac{1}{I_k}=\Theta(\log n)$

Next: random graph

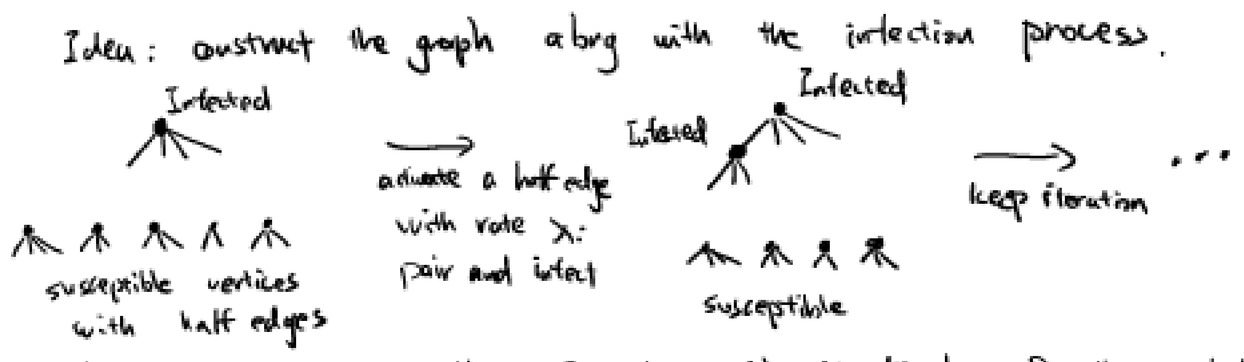
Config model, with nice degree distribution (e.g. compact support)

Each infected $\to$ susceptible neighbor with rate λ.

λ large: outbreak with significant prob.

λ small: no outbreak with high prob.

Idea: construct the graph along with the infection process.



at time t: S_t = # susceptible, $S_{t,k}$ = #susceptible with deg $=k$, B_t = # open half edges in susceptible $=\sum_k k S_{t,k}$

I_t = # Infected, A_t = # open half edges in infected

with rate $\lambda A_t\cdot\frac{kS_{t,k}}{A_t+B_t-1}$, A_t increases by $k-2$; with rate $\lambda A_t\frac{A_t-1}{A_t+B_t-1}$ decreases by 2

on the other hand, each <u>half edge</u> in infected is removed with rate 1

$$\Rightarrow \frac{1}{\delta}\mathbb{E}[A_{t+\delta}-A_t\mid\mathcal{F}_t]=\lambda A_t\cdot\frac{\sum_k k(k-2)S_{t,k}-2(A_t-1)}{A_t+B_t-1}-A_t \quad (*)$$

at small time, i.e. most susceptible, $B_t\simeq n\mathbb{E}D$, $\sum_k k(k-2)S_{t,k}\simeq n\mathbb{E}D(D-2)$ for $D\sim$ deg distribution. ; $A_t\ll B_t$

$\Rightarrow (*)\approx A_t\left(\lambda\cdot\frac{\mathbb{E}D(D-2)}{\mathbb{E}D}-1\right)$; outbreak (with pos. prob) if $\lambda>\frac{\mathbb{E}D}{\mathbb{E}D(D-2)}=\frac{1}{\mathbb{E}D^*-1}$, D^* = biased deg distribution / off spring distribution

<u>Alternative point of view</u>

For a Galton-Watson tree with off spring distribution D^*

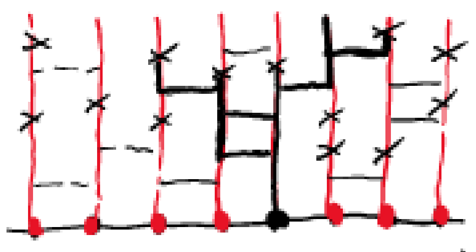
for each child, prob[infected before parent removed] $=\frac{\lambda}{1+\lambda}$

$\Rightarrow \mathbb{E}[\text{\# infected children}\mid\text{parent infected}]=\frac{\lambda}{1+\lambda}\mathbb{E}D^*$

this is >1 iff $\lambda>\frac{1}{\mathbb{E}D^*-1}$.

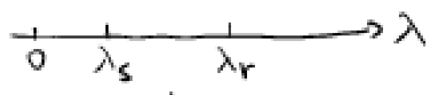
Contact process / SIS model

Only two status: Infected / Susceptible ; each infected $\to$ susceptible with rate 1
each infected infects each susceptible neighbor with rate λ



directed percolation on $V\times\mathbb{R}_+$

d-regular tree: is there an outbreak (infected survives with pos. prob)?
does root get infected as time $\to\infty$ ($\mathbb{P}[\text{root infected at time } t]\to 0$)?



both monotone in λ: two thresholds.

Upper bound: # of infected dominated by the following $(A_t)_{t\ge0}$: $A_0=1$;
at time t, decrease by 1 with rate A_t
increase by 1 with rate $d\lambda A_t$

$\Rightarrow \frac{1}{\delta}\mathbb{E}[A_{t+\delta}-A_t\mid A_t]=(d\lambda-1)A_t$;

die out quickly when $\lambda<\frac{1}{d}$; $\lambda_s\ge\frac{1}{d}$

For λ_r, let $W_t=\sum_{x\in\text{Infected}(t)}\alpha^{-d(\text{root},x)}$

$\Rightarrow \frac{1}{\delta}\mathbb{E}[W_{t+\delta}-W_t\mid\mathcal{F}_t]\le W_t\left(-1+\lambda\alpha+\lambda(d-1)\alpha^{-1}\right)$

Take $\alpha=\sqrt{d-1}$; $\frac{1}{\delta}\mathbb{E}[W_{t+\delta}-W_t\mid\mathcal{F}_t]\le W_t\left(-1+2\sqrt{d-1}\lambda\right)$

$W_t\to0$ quickly when $\lambda<\frac{1}{2\sqrt{d-1}}$; $\lambda_r\ge\frac{1}{2\sqrt{d-1}}$

Lower bound:

For each v with $d(v,\text{root})=k$, let $\mathcal{E}_v$ be the event: infection $v'\to v$ in time $[k-1,k]$ (v': parent of v)
v not recovered in time $[k-1,k+1]$

Then $\mathbb{P}[\mathcal{E}_v]=(1-e^{-\lambda})e^{-2}\approx\lambda e^{-2}$

Think of percolation on tree: each edge opens with prob $(1-e^{-\lambda})e^{-2}$

<u>Pos. prob percolates</u>: $(1-e^{-\lambda})e^{-2}(d-1)>1$; $\lambda_s\lesssim\frac{1}{d}$
$\Leftrightarrow$ infected survives

For λ_r: let $\mathcal{E}_v$ be the event: infection $v'\to v$ in $[k-1,k]$ v not recovered in $[k-1,k+1]$
$v\to v'$ in $[w-k-1,w-k]$ v not recovered in $[w-k-2,w-k]$

Now: percolation with open prob $(1-e^{-\lambda})^2e^{-4}$
percolates to level $\frac{w}{2}$ $\Leftrightarrow$ root infected at time w ; $(1-e^{-\lambda})^2e^{-4}(d-1)>1\Rightarrow\lambda_r\lesssim\frac{1}{\sqrt{d}}$

Some other results:

If $\lambda<\lambda_c$, $\mathbb{E}[\text{\# infected at time } t]\le e^{-ct}$

If $\lambda>\lambda_c$, $\mathbb{E}[\text{\# infected at time } t]\ge e^{ct}$

On random d-reg graph of n vertices: let T be first time with no infected.

If $\lambda<\lambda_c$ $\mathbb{P}[T>C\log n\mid\text{any initial}]\to0$
If $\lambda>\lambda_c$ $\mathbb{P}[T>e^{cn}\mid\text{random initial}]>\varepsilon$ as $n\to\infty$

(References: (Pimentle, 92; Lalley-Su, 15)