Random Matrices II

Saturday, April 12, 2025 2:55 PM

So far: general Wigner / sample covariance matrices

Limit shape: semi-circle / Marchenko-Pastur law

Moment method / Stieltjes transform.

Next: specific classical matrix models.

Gaussian Orthogonal Ensemble / Gaussian Unitary Ensemble GOE/GUE

$\{U_{ij}\}$ $\{V_{ij}\}$ iid $\mathcal{N}(0,1)$

$$\text{GOE: } \begin{bmatrix} \sqrt{2}U_{11} & U_{12} & U_{13} & \cdots & U_{1n} \\ U_{12} & \sqrt{2}U_{22} & U_{23} & & \vdots \\ U_{13} & U_{23} & \sqrt{2}U_{33} & & \vdots \\ \vdots & & & & \\ U_{1n} & \cdots & & & \sqrt{2}U_{nn} \end{bmatrix} \qquad \text{GUE } \begin{bmatrix} U_{11} & \frac{U_{12}+iV_{12}}{\sqrt{2}} & \frac{U_{13}+iV_{13}}{\sqrt{2}} & \cdots & \frac{U_{1n}+iV_{1n}}{\sqrt{2}} \\ \frac{U_{12}+iV_{12}}{\sqrt{2}} & U_{22} & \frac{U_{23}+iV_{23}}{\sqrt{2}} & & \vdots \\ \frac{U_{13}+iV_{13}}{\sqrt{2}} & \frac{U_{23}+iV_{23}}{\sqrt{2}} & U_{33} & & \\ \vdots & & & & \\ \frac{U_{1n}+iV_{1n}}{\sqrt{2}} & \cdots & & & U_{nn} \end{bmatrix}$$

$$\left(\stackrel{d}{=} \frac{1}{\sqrt{2}}(U+U^t)\right) \qquad \left(\stackrel{d}{=} \frac{1}{2}(U+iV+U^t-iV^t)\right)$$

Highly symmetry: density w.r.t to Lebesgue on $\mathbb{R}^{\frac{n(n+1)}{2}}$ or $\mathbb{C}^{\frac{n(n-1)}{2}} \times \mathbb{R}^n = \mathbb{R}^{n^2}$ given by:

GOE: $C \exp\left\{-\sum_{i=1}^n \frac{X_{ii}^2}{4} - \sum_{1\le i<j\le n} \frac{X_{ij}^2}{2}\right\} = C\exp\left\{-\frac{\operatorname{tr}X^2}{4}\right\} = C\exp\left\{-\frac{1}{4}\sum_{i=1}^n \lambda_i^2\right\}$

GUE: $C\exp\left\{-\sum_{i=1}^n \frac{Z_{ii}^2}{2} \sum_{1\le i<j\le n} |Z_{ij}|^2\right\} = C\exp\left\{-\frac{\operatorname{tr}Z^2}{2}\right\} = C\exp\left\{-\frac{1}{2}\sum_{i=1}^n \lambda_i^2\right\}$

- What is the joint eigenvalue distribution?

Need to integrate out the measure of {all matrices with eigenvalue $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$}

i.e. density of eigenvalues at $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n = C\exp\left\{-\frac{\beta}{4}\sum_{i=1}^n \lambda_i^2\right\} \cdot$ "Vol{matrices with eigenvalue $\lambda_1 \ge \cdots \ge \lambda_n$}".

$\beta=1$ for GOE

$\beta=2$ for GUE

Weyl integration formula

If $X=(X_{ij})$ is a real, symmetric random matrix with density $\rho(\lambda_1,\dots,\lambda_n)$ relative to Lebesgue measure $\prod_i dX_{ii}\prod_{i<j}dX_{ij}$ where $\lambda_1,\dots,\lambda_n$ are eigenvalues & ρ is a symmetric function; then the joint density of the eigenvalues relative to Lebesgue measure $\prod d\lambda_i$ is $C_n\,\rho(\lambda_1,\dots,\lambda_n)\prod_{i<j}|\lambda_i-\lambda_j|$

If $Z=(Z_{ij})$ is a complex, Hermitian random matrix with density $\rho(\lambda_1,\dots,\lambda_n)$ ...

...

$\prod d\lambda_i$ is $\tilde{C}_n\,\rho(\lambda_1,\dots,\lambda_n)\prod_{i<j}(\lambda_i-\lambda_j)^2$

- Apply to GOE/GUE: $\rho(\lambda_1,\dots,\lambda_n) \sim \exp\left\{-\frac{\beta}{4}\sum_{i=1}^n\lambda_i^2\right\}$

$\Rightarrow$ eigenvalue distribution $\sim \exp\left\{-\frac{\beta}{4}\sum_{i=1}^n\lambda_i^2\right\}\prod_{i<j}|\lambda_i-\lambda_j|^\beta \prod d\lambda_i$ $\quad\begin{cases}\beta=1 & \text{GOE}\\ \beta=2 & \text{GUE}\end{cases}$

Next: prove Weyl integration formula (real, symmetric matrix case)

Step 1. For any orthonormal matrix O (i.e. $OO^T=I$), $OXO^T \stackrel{d}{=} X$.

proof. Let $\tilde{X}=(\tilde{X}_{ij})=OXO^T$; then $\tilde{X}_{ij}=\sum_{i'j'} O_{ii'}X_{i'j'}O_{jj'}$; $\tilde{X}$ has the same eigenvalues as X.

$L: X\mapsto OXO^T$ a linear transformation in $\mathbb{R}^{\frac{n(n+1)}{2}}$; think of L as a $\frac{n(n+1)}{2}\times\frac{n(n+1)}{2}$ matrix.

Want to show: L preserves Lebesgue measure; $|\det L|=1$

Note: $\sum_{i,j=1}^n \tilde{X}_{ij}^2 = \sum_{i,j=1}^n X_{ij}^2$; $\sum_{i=1}^n \tilde{X}_{ii}^2 + \sum_{i<j}2\tilde{X}_{ij}^2 = \sum_{i=1}^n X_{ii}^2 + \sum_{i<j}2X_{ij}^2$.

$\Rightarrow L$ is a orthonormal transform in $\mathbb{R}^{\frac{n(n+1)}{2}} \Rightarrow |\det L|=1$

Implication: $X\stackrel{d}{=}OXO^T$ for $O\sim$ uniform from $O(n)$ (Haar measure)

Can diagonalize $X=ADA^T$; then $A\stackrel{d}{=}OA$; then $A\sim$ uniform from $O(n)$

(not unique, 2^n possibilities; choose one uniformly random)

Note: columns of A are eigenvectors; eigenvectors $\sim$ uniform from $O(n)$

Step 2. Now that $X=ADA^T$, $A\sim$ unif $O(n)$, need to figure out Jacobi determinant of $(A,D)\mapsto ADA^T=X$

From step 1, this is independent of A; let's compute the derivatives at $A=I_n$.

For $(Q_{ij})_{i,j=1}^n \in \mathbb{R}^{n\times n}$ with $Q_{ij}=-Q_{ji}$, $A=\exp(Q)=\sum_{k=0}^\infty \frac{1}{k!}Q^k \in SO(n)$, gives a coordinate chart of $SO(n)$

can consider $(D,Q)\mapsto \exp(Q)D\exp(-Q)=X$, $D=\begin{pmatrix}\lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n\end{pmatrix}$

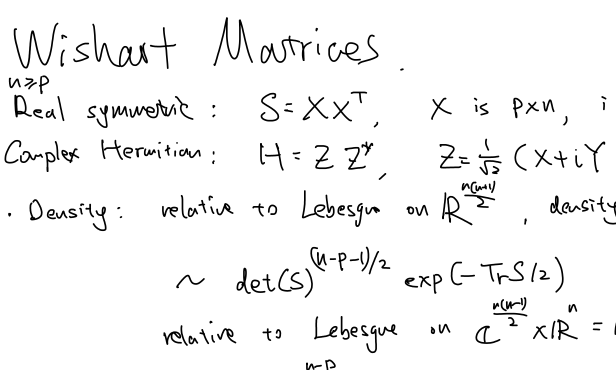
at $Q=0$ $(A=I_n)$, we have

[when $A\approx I_n$ | $Q\approx 0$, $\exp(Q)\approx I+Q$, $\exp(-Q)\approx I-Q$]

$$\frac{\partial X_{ii}}{\partial\lambda_j}=\delta_{ij},\quad \frac{\partial X_{ii}}{\partial Q_{jk}}=0,\quad \frac{\partial X_{ij}}{\partial\lambda_k}\ (i<j)=0,\quad \frac{\partial X_{ij}}{\partial Q_{i'j'}}\ (i<j,\ i'<j')=\delta_{ii'}\delta_{jj'}(\lambda_i-\lambda_j)$$

$\Rightarrow |\det \text{Jacobian}| = \prod_{i<j}|\lambda_i-\lambda_j|$

☆ A side comment: Gaussian corners process



For GUE/GOE, take upper-left corners

array of eigenvalues $(\lambda_i^{(k)})_{1\le i\le k\le n}$

interlacing: $\lambda_i^{(k)} \ge \lambda_i^{(k-1)} \ge \lambda_{i+1}^{(k)}$

prob density $\sim \prod_{1\le i<j\le n}(\lambda_i^{(n)}-\lambda_j^{(n)})\prod_{k=1}^{n-1}\prod_{1\le i<j\le k}(\lambda_i^{(k)}-\lambda_j^{(k)})^{2-\beta}\prod_{i=1}^k\prod_{j=1}^{k+1}|\lambda_i^{(k)}-\lambda_j^{(k+1)}|^{\beta/2-1}\cdot\exp\left(-\frac{\beta}{4}\sum_{i=1}^n(\lambda_i^{(n)})^2\right)$

(For GUE / $\beta=2$, this is $\prod_{1\le i<j\le n}(\lambda_i^{(n)}-\lambda_j^{(n)})\cdot\exp\left(-\frac{1}{2}\sum_{i=1}^n(\lambda_i^{(n)})^2\right)$; uniform given top level)

Wishart Matrices

$n\ge p$

Real symmetric: $S=XX^T$, X is $p\times n$, iid $\mathcal{N}(0,1)$

Complex Hermitian: $H=ZZ^*$, $Z=\frac{1}{\sqrt{2}}(X+iY)$, X,Y $p\times n$, iid $\mathcal{N}(0,1)$

- Density: relative to Lebesgue on $\mathbb{R}^{\frac{n(n+1)}{2}}$, density of S is

$\sim \det(S)^{(n-p-1)/2}\exp(-\operatorname{Tr}S/2)$

relative to Lebesgue on $\mathbb{C}^{\frac{n(n-1)}{2}}\times\mathbb{R}^n=\mathbb{R}^{n^2}$, density of H is

$\sim \det(H)^{n-p}\exp(-\operatorname{Tr}H)$

Using Weyl integration formula

eigenvalue density of $S \sim \prod_{i=1}^p \lambda_i^{(n-p-1)/2}e^{-\lambda_i/2}\prod_{i<j}(\lambda_i-\lambda_j)$ (Laguerre Orthogonal/Unitary Ensemble)

– – – of $H \sim \prod_{i=1}^p \lambda_i^{(n-p)}e^{-\lambda_i}\prod_{i<j}(\lambda_i-\lambda_j)^2$ LOE / LUE

Proof of density:

(real symmetry case)

Compute the moment generating function of S

i.e. take real symmetric $R\in\mathbb{R}^{p\times p}$

$\mathbb{E}\exp\left\{\sum_{i,j=1}^p R_{ij}S_{ij}\right\} = \mathbb{E}\exp(\operatorname{Tr}RS) = \mathbb{E}\exp(\operatorname{Tr}X^TRX)$

diagonalize: $R=ADA^T$; $A^TX\stackrel{d}{=}X$

$\Rightarrow = \mathbb{E}\exp(\operatorname{Tr}X^TDX) = \mathbb{E}\exp\left\{\sum_{i=1}^n\sum_{j=1}^p X_{ij}^2 d_{jj}\right\} = \prod_{i=1}^p(1-2d_{ii})^{-\frac{n}{2}}$, assuming each $d_{ii}<\frac{1}{2}$

$= \det(I-2D)^{-n/2} = \det(I-2R)^{-n/2}$

On the other hand, for the density $\sim \det(S)^{(n-p-1)/2}\exp(-\operatorname{Tr}S/2)$

$\int \det(S)^{(n-p-1)/2}\exp(-\operatorname{Tr}S/2)\cdot\exp(\operatorname{Tr}RS)\,dS$ (invariance under conjugation)

$=\int\det(S)^{(n-p-1)/2}\exp(-\operatorname{Tr}S/2)\cdot\exp(\operatorname{Tr}DS)\,dS$

$=\int\det(S)^{(n-p-1)/2}\exp(-\operatorname{Tr}((I_p-2D)S/2))\,dS$

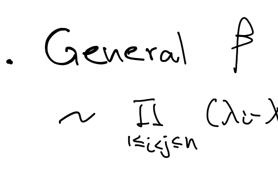
rescale by $\tilde{S}_{ij}=\sqrt{(1-2d_{ii})(1-2d_{jj})}\,S_{ij}$; $d\tilde{S}=(\det(I-2D))^{\frac{p+1}{2}}dS$, $\det(\tilde{S})=\det(I-2D)\det(S)$

$=\det(I-2D)^{-n/2}\int\det(\tilde{S})^{(n-p-1)/2}\exp(-\operatorname{Tr}\tilde{S}/2)\,d\tilde{S}$

$=\det(I-2R)^{-n/2}$ – – –

MANOVA matrices

(Multivariate analysis of variance)



X: $n\times p$ iid $\mathcal{N}(0,1)$ / $\frac{1}{\sqrt{2}}(\mathcal{N}(0,1)+\mathcal{N}(0,1))$

Y: $n\times q$ iid " "

$(p,q\ge n+1)$

$M = XX^*(XX^*+YY^*)^{-1}$

(i.e. using 2 Wishart matrices)

$\Rightarrow$ density $\sim \det(M)^{\frac{\beta}{2}(|n-p|+1)-1}\det(I-M)^{\frac{\beta}{2}(|n-q|+1)-1}$

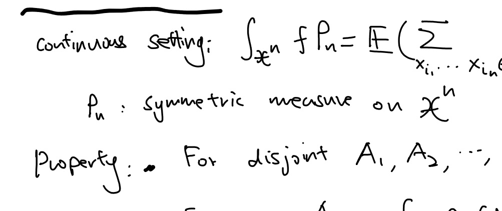
$\Rightarrow$ eigenvalue density $\sim \prod_{1\le i<j\le n}(\lambda_i-\lambda_j)^\beta\cdot\prod_{i=1}^n\lambda_i^{\frac{\beta}{2}(|n-p|+1)-1}(1-\lambda_i)^{\frac{\beta}{2}(|n-q|+1)-1}$ Jacobi Orthogonal/Unitary Ensemble

- General β ensemble (on $W_n=\{(\lambda_1,\dots,\lambda_n):\lambda_1\ge\lambda_2\ge\cdots\ge\lambda_n\}$)

$\sim\prod_{1\le i<j\le n}(\lambda_i-\lambda_j)^\beta\prod_{i=1}^n V(\lambda_i)$

Limiting shape differs: semi-circle, Marchenko-Pastur, etc (depends on V)

Local law: universal



same for all of them!

(scaling parameters might differ)

Determinantal Point Process

A point process is a random subset $X\in\mathcal{X}=$ countable discrete, e.g. $\mathbb{Z}^d$; continuous, e.g. $\mathbb{R}^d$

Correlation function: $\rho(A)=\mathbb{P}(A\subseteq X)$ discrete setting

continuous setting: $\int_{\mathcal{X}^n} f\rho_n = \mathbb{E}\left(\sum_{x_{i_1},\dots,x_{i_n}\in X} f(x_{i_1},\dots,x_{i_n})\right)$ for any f on $\mathcal{X}^n$, compactly supported

ρ_n: symmetric measure on $\mathcal{X}^n$ (also use ρ_n to denote is density)

Property: • For disjoint $A_1,A_2,\dots,A_n$, $\int_{A_1\times\cdots\times A_n}\rho_n(dx_1,\dots,dx_n)=\mathbb{E}[|A_1\cap X|\cdots|A_n\cap X|]$

• For any A, $\int_{A^n}\rho_n(dx_1,\dots,dx_n)=\mathbb{E}[|A\cap X|(|A\cap X|-1)(|A\cap X|-2)\cdots(|A\cap X|-n+1)]$

Example: Poisson point process on $\mathbb{R}$ with rate $\lambda \Rightarrow \rho_n\equiv\lambda^n$

(For $X\sim\mathrm{Poi}(a\lambda)$, $\mathbb{E}X(X-1)\cdots(X-n+1)=\sum_{k=0}^\infty e^{-a\lambda}\cdot\frac{(a\lambda)^k}{k!}k(k-1)\cdots(k-n+1)=(a\lambda)^n=\int_{[0,a]^n}\lambda^n\,dx_1\cdots dx_n$)

A point process on $\mathcal{X}$ is determinantal, if there exists K on $\mathcal{X}\times\mathcal{X}$ s.t. for each $n\in\mathbb{N}$,

$\rho_n(x_1,\dots,x_n)=\det[K(x_i,x_j)]_{i,j=1}^n$

Such K is the correlation kernel

K is not necessarily unique: $K(x,y)\cdot\frac{f(x)}{f(y)}$ also a kernel.

Example: Poisson point process $K(x,y)=\lambda\cdot\delta_{xy}$

The process is Fermion if $K(x,y)=\overline{K}(y,x)$; in this case $\rho_2(x,y)=\rho_1(x)\rho_1(y)-|K(x,y)|^2\le\rho_1(x)\rho_1(y)$

$\Rightarrow$ particles repel

Next: determinantal structure of GUE

Recall the density of $\lambda_1\ge\lambda_2\ge\cdots\ge\lambda_n$ given by

$\sim\prod_{i<j}(\lambda_i-\lambda_j)^2\prod_{i=1}^n\exp(-\lambda_i^2/2)$ $(*)$

up to a constant.

Vandermonde matrix:

$$V=\begin{pmatrix}1 & 1 & 1 & \cdots & 1\\ \lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_n\\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \cdots & \lambda_n^2\\ & & \vdots & & \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \lambda_3^{n-1} & \cdots & \lambda_n^{n-1}\end{pmatrix}$$

Claim: $\det V=\prod_{i<j}(\lambda_j-\lambda_i)$

proof $\det V$ is a polynomial of $\deg=\frac{n(n-1)}{2}$; contains a factor of $\lambda_j-\lambda_i$ for each $1\le i<j\le n$,

$\Rightarrow \det V=C\prod_{i<j}(\lambda_j-\lambda_i)$; $C=1$ by considering the coefficient of $\lambda_2\lambda_3^2\cdots\lambda_n^{n-1}$

$\Rightarrow (*)\sim\left(\det[P_{i-1}(\lambda_j)\exp(-\lambda_j^2/4)]_{i,j=1}^n\right)^2=\det\left[\sum_{k=0}^{n-1}P_k(\lambda_i)P_k(\lambda_j)\exp(-(\lambda_i^2+\lambda_j^2)/4)\right]_{i,j=1}^n$

$P_0,\dots,P_{n-1}$ are any polynomials of deg $0,\dots,n-1$

Let $K_n(x,y)=\sum_{k=0}^{n-1}\psi_k(x)\psi_k(y)\exp(-(x^2+y^2)/4)$, ψ_k Hermite polynomials: $\int\psi_k(x)\psi_j(x)e^{-x^2/2}dx=\delta_{kj}$

$\Rightarrow (*)\sim\det[K_n(\lambda_i,\lambda_j)]_{i,j=1}^n$

property: $\int K_n(x,x)\,dx=n$

$\int K_n(x,y)K_n(y,z)\,dy=K_n(x,z)$

Lemma. For any $k\ge 0$,

$\int\det[K_n(x_i,x_j)]_{i,j=1}^{k+1}dx_{k+1}=(n-k)\det[K_n(x_i,x_j)]_{i,j=1}^k$

proof LHS $=\int\sum_{\sigma\in S_{k+1}}\operatorname{sgn}(\sigma)\cdot\prod_{i=1}^{k+1}K_n(x_i,x_{\sigma(i)})\,dx_{k+1}=\sum_{\substack{\sigma\in S_{k+1}\\ \sigma(k+1)=k+1}}\operatorname{sgn}(\sigma)\cdot n\cdot\prod_{i=1}^{k+1}K_n(x_i,x_{\sigma(i)})-\sum_{\substack{\sigma\in S_{k+1}\\ \sigma(k+1)\ne k+1}}\operatorname{sgn}(\sigma)\cdot\prod_{i=1}^kK_n(x_i,x_{\tilde{\sigma}(i)})$

$\tilde{\sigma}:\begin{cases}\tilde{\sigma}(i)=\sigma(i) & \text{if } \sigma(i)\ne k+1\\ \tilde{\sigma}(i)=\sigma(\sigma(i))=\sigma(k+1) & \text{if } \sigma(i)=k+1\end{cases}$

$=\sum_{\tilde{\sigma}\in S_k}(n-k)\operatorname{sgn}(\tilde{\sigma})\prod_{i=1}^kK_n(x_i,x_{\tilde{\sigma}(i)})$

$=(n-k)\det[K_n(x_i,x_j)]_{i,j=1}^k$

In particular. $\int_{\mathbb{R}^k}\det[K_n(x_i,x_j)]_{i,j=1}^n dx_1,\dots,dx_n=n!$

$\Rightarrow \rho_n(x_1,x_2,\dots,x_n)=\det[K_n(x_i,x_j)]_{i,j=1}^n$, since $\int_{\mathbb{R}^n}\rho_n(x_1\cdots x_n)\,dx_1\cdots dx_n=n!$

Also $\rho_k(x_1,x_2,\dots,x_k)=\det[K_n(x_i,x_j)]_{i,j=1}^k$, since $\int_{\mathbb{R}^k}\rho_n(x_1,\dots x_k)\,dx_1\cdots dx_k=\int_{\mathbb{R}^k}\det[K_n(x_i,x_j)]_{i,j=1}^k dx_1\cdots dx_k=n\cdots(n-k+1)$

$\Rightarrow\{\lambda_1,\dots,\lambda_n\}$ is a det. point. process with kernel K_n.