Random Matrices III

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Bulk / Edge limits of GUE

Preperation: for determinant point process, what is convergence?

For a sequence of point process $X^{(1)}, X^{(2)}, \cdots$ on $\mathbb{R}$, they $\to X$, if $\sum_{x\in X^{(n)}} \delta_x \to \sum_{x\in X}\delta_x$ weakly in vague topology,

(i.e. $\sum_{x\in X^{(n)}} f(x) \to \sum_{x\in X} f(x)$ in distribution, $\forall f\in C_c$, jointly for finitely many f)

Assume X has no delta mass, i.e. $\mathbb{P}[x\in X]=0$ for any $x\in\mathbb{R}$, this is equivalent to

$(\# X^{(n)}\cap I_1, \# X^{(n)}\cap I_2, \cdots, \# X^{(n)}\cap I_k) \longrightarrow (\# X\cap I_1, \# X\cap I_2, \cdots, \# X\cap I_k)$ jointly, for any disjoint intervals $I_1, \cdots, I_k$ $(*)$

For determinantal point precess with kernel $K: \mathbb{R}^2\to\mathbb{C}$, usually think of "trace-class convergence".

Some functional analysis: consider the Hilbert space $H = L^2(\mathbb{R})$, a bounded linear operator $T: H\to H$ is in "trace-class", if

$\mathrm{Tr}|T| = \sum_{k=1}^{\infty}\langle |T|e_k, e_k\rangle < \infty$, $|T| := \sqrt{T^*T}$ positive-semi-def Hermitian sq.root ($T=|T|$ if T Hermitian & pos-semi-def)

Properties: $\|T\|_1 := \mathrm{Tr}|T|$ is a norm (trace norm), $\mathrm{Tr}\,T \le \|T\|_1$ (Lidskii)

If T is trace class, $A: H\to H$ bounded $\Rightarrow AT, TA$ are trace class.

Can define the Fredholm determinant $\det(I+T) = \sum_{k=0}^{\infty}\mathrm{tr}\underbrace{\Lambda^k T}_{\text{acting on }\Lambda^k H}$ $\left(\sum_{k=0}^{\infty}|\mathrm{tr}\Lambda^k T| \le \sum_{k=0}^{\infty}\|\Lambda^k T\| \le \sum_{k=0}^{\infty}\frac{\|T\|_1^k}{k!} \le e^{\|T\|_1}\right)$

can think of as $\det(I+T) = \prod_{k=1}^{\infty}(1+\lambda_k)$; $\sum_{k=1}^{\infty}|\lambda_k| \le \|T\|_1 < \infty$ ensures that the product is finite.

For $K: \mathbb{R}^2\to\mathbb{C}$, say Fermion, i.e. $K(x,y) = \overline{K(y,x)}$, gives an operator via $Kf(x) = \int K(x,y)f(y)\,dy$

(Suppose K pos-semi-def, $K=|K|$, $\|K\|_1 = \mathrm{Tr}|K| = \mathrm{Tr}\,K = \int K(x,x)\,dx$. finite $\Rightarrow$ K Fredholm)

$\det(I+K) = \sum_{n=0}^{\infty}\frac{1}{n!}\int \det[K(x_i,x_j)]_{i,j=1}^n\, dx_1\cdots dx_n$; converges if K is Fredholm.

Probability meaning for determinantal point process with kernel K,

$\det(\mathrm{Id} - KP_I) = 1 - \mathbb{E}[\# I\cap X] + \frac{1}{2!}\mathbb{E}[(\# I\cap X)(\# I\cap X - 1)] - \frac{1}{3!}\mathbb{E}[(\# I\cap X)(\# I\cap X-1)(\# I\cap X-2)] + \cdots = \mathbb{P}[\# I\cap X = 0]$

($I\subseteq\mathbb{R}$ a subset, $P_I f(x) = \mathbb{1}[x\in I]f(x)$)

More generally: for disjoint intervals $I_1, \cdots, I_k$, and $z_1, \cdots, z_k\in\mathbb{C}$

$\det\left(\mathrm{Id} - \sum_{i=1}^{k}(1-z_i)KP_{I_i}\right) = \underbrace{\mathbb{E}\left(\prod_{i=1}^{k} z_i^{\# I_i\cap X}\right)}_{\text{gives the joint distribution of } \# I_1\cap X, \cdots, \# I_k\cap X}$

Another property (of Fredholm determinant)

$|\det(I+T) - \det(I+S)| \le \|T-S\|_1 \exp(\max(\|T\|_1, \|S\|_1)+1)$

$\Rightarrow$ $T^{(1)}, T^{(2)}, \cdots \to T$ in trace class, implies that $\det(\mathrm{Id}+T^{(1)}), \det(\mathrm{Id}+T^{(2)}) \to \det(\mathrm{Id}+T)$

For a sequence of determinant point processes (in $\mathbb{R}$), with Hermitian and pos-semi-def kernel, suffices to prove converge in trace class. (of compact interval projections)

(proof: via functional analysis, or compute the determinants directly using

$|\det(\mathrm{Id}-KP_I) - \det(\mathrm{Id}-\tilde{K}P_I)| \le \sum_{n=1}^{\infty}\frac{n^{(n+1)/2}}{n!}|I|^n\cdot\max\{\|\tilde{K}\|_\infty, \|K\|_\infty\}^{n-1}\cdot\|K-\tilde{K}\|_\infty$)

Now back to GUE

$K^{(n)}(x,y) = \sum_{k=0}^{n-1}\psi_k(x)\psi_k(y)\exp\{-\frac{x^2}{4}-\frac{y^2}{4}\}$, $\psi_0, \cdots, \psi_{n-1}$ Hermite polynomials. ($\int\psi_i(x)\psi_j(x)e^{-x^2/2}dx = \delta_{ij}$)

In light of semi-circle law: $\frac{1}{\sqrt{n}}K^{(n)}(x\sqrt{n}, x\sqrt{n}) \approx \rho_{sc}(x) = \frac{1}{2\pi}\sqrt{4-x^2}$, for $x\in[-2,2]$

More precise limiting law

① Bulk: fix $x\in(-2,2)$, $\frac{1}{\sqrt{n}\rho_{sc}(x)}K^{(n)}\left(x\sqrt{n}+\frac{y}{\sqrt{n}\rho_{sc}(x)}, x\sqrt{n}+\frac{z}{\sqrt{n}\rho_{sc}(x)}\right) \longrightarrow \frac{\sin(\pi(y-z))}{\pi(y-z)}$, unif. in compact for y,z

② Edge: $\frac{1}{n^{1/6}}K^{(n)}\left(2\sqrt{n}+\frac{y}{n^{1/6}}, 2\sqrt{n}+\frac{z}{n^{1/6}}\right) \to K_{Ai}(y,z) = \frac{Ai(y)Ai'(z)-Ai'(y)Ai(z)}{y-z}$; Ai is Airy function, i.e. $A''(x) = xAi(x)$

$[K_{Ai}(y,y) = Ai'(y)^2 - yAi(y)^2]$ det. point. pro. with K_{Ai} is called Airy point process

Heuristic explanation:

Let $\phi_k(x) = \psi_k(x)\exp\{-\frac{x^2}{4}\}$, then $\phi_0, \phi_1, \cdots$ orthonormal ($\int\phi_i\phi_j = \delta_{ij}$)

consider operator $L: -\frac{d^2}{dx^2}+\frac{x^2}{4}$; $L\phi_k = (k+\frac{1}{2})\phi_k$; $\Rightarrow \phi_0, \phi_1, \cdots$ eigenbasis of L

$K^{(n)}$: projection onto the span of $\phi_0, \phi_1, \cdots, \phi_{n-1}$; or can write $K^{(n)} = \mathbb{1}_{(-\infty, n-\frac{1}{2}]}[L]$

spectral projection of operators in $L^2(\mathbb{R})$

① Bulk: rescale L, by taking $x = x_0\sqrt{n}+\frac{s}{\sqrt{n}\rho_{sc}(x_0)}$;

$\Rightarrow \mathbb{1}_{(-\infty, n-\frac{1}{2}]}\left[-n\rho_{sc}(x_0)^2\frac{d^2}{ds^2}+\frac{1}{4}\left(x_0\sqrt{n}+\frac{s}{\sqrt{n}\rho_{sc}(x_0)}\right)^2\right]$

approximately: $\mathbb{1}_{(-\infty, n]}\left[-n\rho_{sc}(x_0)^2\frac{d^2}{ds^2}+\frac{n}{4}x_0^2\right] = \mathbb{1}_{(-\infty,\pi^2]}\left[-\frac{d^2}{ds^2}\right]$

Spectrum / "eigenfunction" of $-\frac{d^2}{ds^2}$? $e^{2\pi i\theta s}$ has "eigenvalue" $4\pi^2\theta^2$

$\Rightarrow$ Fourier projection onto $\theta\in[-\frac{1}{2},\frac{1}{2}]$

$Pf(s) = \int_{\mathbb{R}}\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i\theta s}e^{-2\pi i\theta t}\,d\theta\, f(t)\,dt = \int_{\mathbb{R}}\frac{\sin(\pi(s-t))}{\pi(s-t)}f(t)\,dt$

② Edge: by taking $x = 2\sqrt{n}+\frac{s}{n^{1/6}}$, we get

$\mathbb{1}_{(-\infty, n-\frac{1}{2}]}\left[-n^{\frac{1}{3}}\frac{d^2}{ds^2}+\frac{1}{4}\left(2\sqrt{n}+\frac{s}{n^{1/6}}\right)^2\right] \approx \mathbb{1}_{(-\infty,n]}\left[-n^{\frac{1}{3}}\frac{d^2}{ds^2}+n+sn^{\frac{1}{3}}\right] = \mathbb{1}_{(-\infty,0]}\left[-\frac{d^2}{ds^2}+s\right]$

For such Pf, take Fourier transform $\hat{f}(\theta) := \int e^{-2\pi i\theta s}f(s)\,ds$, then $\hat{P}\hat{f}$ with $\hat{P} = \mathbb{1}_{(-\infty,0]}\left[4\pi^2\theta^2+\frac{1}{2\pi i}\frac{d}{d\theta}\right]$

Let $\tilde{f}(\theta) = e^{8\pi^3 i\theta^3/3}\hat{f}(\theta)$, consider $\tilde{P}\tilde{f}$ with $\tilde{P} = \mathbb{1}_{(-\infty,0]}\left[\frac{d}{d\theta}\right]$

Let $\bar{f}(t) = (\tilde{f})^\vee$ (inverse Fourier), $\bar{P} = \mathbb{1}_{(-\infty,0]}[t]$, this is just the operator of multiplying $\mathbb{1}_{(-\infty,0]}$

In summary: original $P = M^{-1}\mathbb{1}_{(-\infty,0]}M$, where

$Mf(t) = \iint e^{2\pi it\theta}e^{8\pi^3 i\theta^3/3}e^{-2\pi is\theta}\,d\theta\, f(s)\,ds = \int Ai(t-s)f(s)\,ds$ ← via another definition of Ai

similarly, $M^{-1}f(t) = \int Ai(s-t)f(s)\,ds$

(observe that $M^{-1} = M^*$)

$\Rightarrow Pf(t) = \int_{\mathbb{R}}\underbrace{\int_{(-\infty,0]}Ai(t-w)Ai(s-w)\,dw}_{}\, f(s)\,ds$

$= \frac{Ai(t)Ai'(s)-Ai(s)Ai'(t)}{t-s} = K_{Ai}(t,s)$

↓ Christoffel-Darboux formula

(proof: $\frac{d}{dw}\frac{Ai(t-w)Ai'(s-w)-Ai(s-w)Ai'(t-w)}{t-s} = Ai(t-w)Ai(s-w)$)

Steepest descent method

Idea: write $K^{(n)}$ as contour integral

Let $P_n = h_n\psi_n$ be the monic Hermite polynomial; then can write $P_n(x) = (-1)^n e^{x^2/2}\frac{d^n}{dx^n}e^{-x^2/2}$ (proof: check orthogonality)

Then $\sum_{n=0}^{\infty}P_n(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty}e^{x^2/2}\frac{(-t)^n}{n!}\frac{d^n}{dx^n}e^{-x^2/2} = e^{x^2/2}e^{-(x-t)^2/2} = e^{xt-t^2/2}$, for any $t\in\mathbb{C}$

Cauchy integral formula: $P_n(x) = \frac{n!}{2\pi i}\oint_C\frac{\exp(xt-t^2/2)}{t^{n+1}}dt$, C encircling the origin.

Another form: $\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp(-\frac{t^2}{2}+itx)dt = e^{-x^2/2}$

$\Rightarrow \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}(it)^n\exp(-\frac{t^2}{2}+itx)dt = \frac{d^n}{dx^n}e^{-x^2/2}$

$\Rightarrow P_n(x) = \frac{(-1)^n e^{x^2/2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(it)^n\exp(-\frac{t^2}{2}+itx)dt$

Let $s = -it$

$\Rightarrow P_n(x) = \frac{ie^{x^2/2}}{\sqrt{2\pi}}\int_{-i\infty}^{i\infty}s^n\exp(\frac{s^2}{2}-sx)\,ds$

Need also find h_n

$h_n^2 = \int_{-\infty}^{\infty}P_n(x)^2e^{-x^2/2}dx = n!\sqrt{2\pi}$

proof. $\exp(xt+xs-\frac{t^2}{2}-\frac{s^2}{2}) = \sum_{m,n=0}^{\infty}P_n(x)P_m(x)\frac{t^n s^m}{n!m!}$

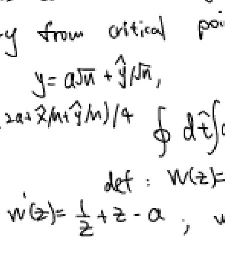
$\Rightarrow \underbrace{\int\exp(-x^2/2+x(t+s)-\frac{t^2}{2}-\frac{s^2}{2})\,dx}_{=\sqrt{2\pi}\exp(ts)} = \sum_{n=0}^{\infty}\frac{h_n^2(ts)^n}{(n!)^2}$

$\Rightarrow$ Taylor expansion in ts gives the conclusion.

Therefore, $K^{(n)}(x,y) = \sum_{k=0}^{n-1}\psi_k(x)\psi_k(y)e^{-\frac{x^2}{4}-\frac{y^2}{4}} = \sum_{k=0}^{n-1}\frac{1}{n!\sqrt{2\pi}}P_k(x)P_k(y)e^{-\frac{x^2}{4}-\frac{y^2}{4}}$

$= \frac{e^{-\frac{x^2+y^2}{4}}}{(2\pi)^2}\oint_C dt\int_{-i\infty}^{i\infty}ds\,\exp(-\frac{t^2}{2}+xt+\frac{s^2}{2}-ys)\underbrace{\sum_{k=0}^{n-1}s^k t^{-k-1}}_{=\frac{1-(s/t)^n}{t-s}}$

deform contours



$= \frac{e^{-\frac{x^2+y^2}{4}}}{(2\pi)^2}\oint_C dt\int_\gamma ds\,\exp(-\frac{t^2}{2}+xt+\frac{s^2}{2}-ys)\underbrace{\sum_{k=-\infty}^{n-1}s^k t^{-k-1}}_{=\frac{(s/t)^n}{t-s}}$ (extra terms have no residue at $t=0$)

Idea: for $n\to\infty$,

deform the contours so that the integrand decays fast away from critical points.

① Bulk: $x = a\sqrt{n}+\hat{x}/\sqrt{n}$, $y = a\sqrt{n}+\hat{y}/\sqrt{n}$, $a\in(-2,2)$, $s = \hat{s}\sqrt{n}$, $t = \hat{t}\sqrt{n}$

$K^{(n)}(x,y) = \frac{\sqrt{n}}{(2\pi)^2}e^{-(\hat{x}-\hat{y})(2a+\hat{x}/n+\hat{y}/n)/4}\oint d\hat{t}\int d\hat{s}\,\frac{1}{\hat{t}-\hat{s}}\exp\left(n\underbrace{\left(\log\hat{s}-\log\hat{t}+\frac{\hat{s}^2}{2}-\frac{\hat{t}^2}{2}+a(\hat{t}-\hat{s})\right)}_{w(\hat{s})-w(\hat{t})}+\frac{\hat{t}\hat{x}-\hat{s}\hat{y}}{n}\right)$

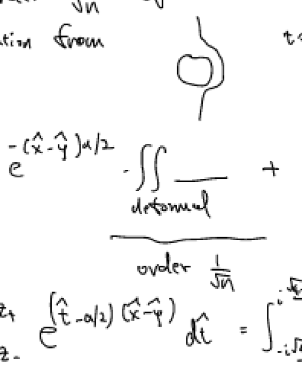
def: $W(z) = \log z+\frac{z^2}{2}-az$

Let's analyze W: $w'(z) = \frac{1}{z}+z-a$; $w'(z)=0$ for $z = \frac{a\pm i\sqrt{4-a^2}}{2} =: z_\pm$ (critical points)

Also $w''(z) = -\frac{1}{z^2}+1$

$\Rightarrow$ near z_+,

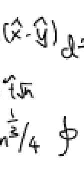
$w(z)\approx w(z_+)+\frac{1}{2}(-\frac{1}{z_+^2}+1)(z-z_+)^2$



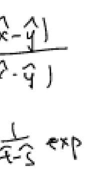
deform contours to pass through z_+ and z_- $\begin{cases}\hat{s}\text{ contour has } Re(w)\downarrow \text{ fast away from } z_+/z_- \\ \hat{t}\text{ contour} \cdots\cdots \uparrow \cdots\cdots\end{cases}$

$\Rightarrow$ only order $\frac{1}{\sqrt{n}}$ neighborhood near z_+/z_- matter. (others exp. small)

Issue: deformation from



to



can cause residues.

$\Rightarrow K^{(n)} = \frac{\sqrt{n}}{(2\pi)^2}e^{-(\hat{x}-\hat{y})a/2}\left(\underbrace{\iint_{\text{deformed}}}_{\text{order }\frac{1}{\sqrt{n}}} + \underbrace{\int\text{residue}}_{\text{order }1}\right) \approx \int_{z_-}^{z_+}e^{\hat{t}(\hat{x}-\hat{y})}\,d\hat{t}$

$\Rightarrow \int_{z_-}^{z_+}e^{(\hat{t}-a/2)(\hat{x}-\hat{y})}\,d\hat{t} = \int_{-i\frac{\sqrt{4-a^2}}{2}}^{i\frac{\sqrt{4-a^2}}{2}}e^{\hat{t}(\hat{x}-\hat{y})}\,d\hat{t}$

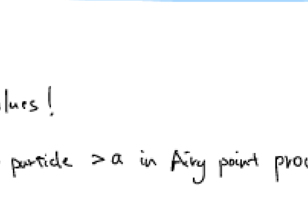
(using $\frac{1}{2\pi i}\int_{-i}^{i}e^{z(\hat{x}-\hat{y})}dz = \frac{\sin(\hat{x}-\hat{y})}{\pi(\hat{x}-\hat{y})}$, get Ksine)

② Edge $x = 2\sqrt{n}+\hat{x}n^{-\frac{1}{6}}$, $y = 2\sqrt{n}+\hat{y}n^{-\frac{1}{6}}$, $s = \hat{s}\sqrt{n}$, $t = \hat{t}\sqrt{n}$

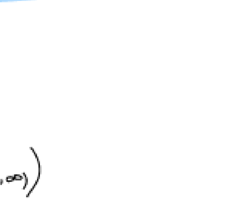
$K^{(n)}(x,y) = \frac{\sqrt{n}}{(2\pi)^2}e^{-(\hat{x}-\hat{y})(4+\hat{x}n^{-\frac{2}{3}}+\hat{y}n^{-\frac{2}{3}})n^{\frac{1}{3}}/4}\oint d\hat{t}\int d\hat{s}\,\frac{1}{\hat{t}-\hat{s}}\exp\left(n\left(\log\hat{s}-\log\hat{t}+\frac{\hat{s}^2}{2}-\frac{\hat{t}^2}{2}+2(\hat{t}-\hat{s})\right)+(\hat{t}\hat{x}-\hat{s}\hat{y})n^{\frac{1}{3}}\right)$

For $W(z) = \log z+\frac{z^2}{2}-2z$, $w'(z) = \frac{1}{z}+z-2$; $w'(1)=0$; $w''(z) = -\frac{1}{z^2}+1$, $w''(1)=0$; $w'''(z) = \frac{2}{z^3}$, $w'''(1)=2$.

$\Rightarrow$ near 1, $w(z)\approx w(1)+\frac{1}{3}(z-1)^3$



deform contours s.t.



only $n^{-\frac{1}{3}}$ neighborhood matters

Let $\hat{s} = 1+u\cdot n^{-\frac{1}{3}}$, $\hat{t} = 1+vn^{-\frac{1}{3}}$

$K^{(n)}(x,y) = \frac{n^{\frac{1}{6}}}{(2\pi)^2}\oint dv\int du\,\frac{1}{v-u}\exp\left(\frac{1}{3}(u^3-v^3)+v\hat{x}-u\hat{y}\right)(1+o(1))$

$= n^{\frac{1}{6}}K_{Ai}(\hat{x},\hat{y}) = n^{\frac{1}{6}}\frac{Ai(\hat{x})Ai'(\hat{y})-Ai'(\hat{x})Ai(\hat{y})}{\hat{x}-\hat{y}}$ (idea of proof: using that $Ai(x) = \frac{1}{2\pi}\int_{-i\infty}^{i\infty}\exp(-\frac{v^3}{3}+vx)\,dv$, then integration by parts)

Further comments

- At edge, actually gets convergence of top eigenvalues!

  $\mathbb{P}[n^{\frac{1}{6}}(\lambda_1^{(n)}-2\sqrt{n}) < a] = \mathbb{P}[\text{no eigenvalue} > 2\sqrt{n}+an^{-\frac{1}{6}}] \to \mathbb{P}[\text{no particle} > a \text{ in Airy point process}] = \det(\mathrm{Id}-K_{Ai}P_{(a,\infty)})$

  More generally, $\{n^{\frac{1}{6}}(\lambda_i^{(n)}-2\sqrt{n})\}_{i=1}^k \longrightarrow \{\alpha_i\}_{i=1}^k$ top k points in Airy point process; joint distribution can be written out using K_{Ai}.

  The top particle distribution, i.e., CDF $F_2(a) = \det[\mathrm{Id}-K_{Ai}P_{(a,\infty)}]$, is known as Tracy-Widom GUE / Tracy-Widom$_2$ distribution.

  (as another expression: $F_2(a) = \exp\left(\int_a^{\infty}(x-s)q^2(x)\,dx\right)$, $q''(s) = sq(s)+2q(s)^3$)

- For GOE, i.e. $\beta=1$ of $\prod_{i<j}|\lambda_i-\lambda_j|^\beta\prod_{i=1}^{n}\exp(-\frac{\beta}{4}\lambda_i^2)$, not determinantal, but Pfaffian (also for $\beta=4$, GSE ↓ symplectic)

  analyze Pfaffian kernel gives similar set of results.

- The same edge / bulk limits also for general $\prod_{i<j}(\lambda_i-\lambda_j)^\beta\prod_{i=1}^{n}V(\lambda_i)$, e.g. Wishart, MANOVA,

  (Note: for edge, need be "soft edge", i.e. no hard constraints $\Rightarrow$ sq root growth of density

  ["Hard edge": Bessel$_{\beta,a}$ process])

- General β: less structure; tridiagonal matrix representation (Dumitriu-Edelman, Ramirez-Rider-Virag)

$$\begin{pmatrix} N(0,2) & \chi_\beta & & & \\ \chi_\beta & N(0,2) & \chi_{2\beta} & & \\ & \chi_{2\beta} & N(0,2) & & \\ & & & \ddots & \chi_{(n-1)\beta} \\ & & & \chi_{(n-1)\beta} & N(0,2)\end{pmatrix} \xrightarrow{\text{at edge}} H_\beta = -\frac{d^2}{dx^2}+x+\frac{2}{\beta}\xi(x)$$

$\xi(x)$ ↓ white noise; dB

$\Rightarrow$ top eigenvalues $\to$ bottom eigenvalues of H_β (lowest one: Tracy-Widom$_\beta$ distribution; formula in terms of Brownian motion)

Tails: $\mathbb{P}[TW_\beta > a] \sim \exp(-\frac{2}{3}\beta a^{3/2})$

$\mathbb{P}[TW_\beta < -a] \sim \exp(-\frac{1}{24}\beta a^3)$ as $a\to\infty$.

at bulk $\longrightarrow$ Sine$_\beta$ process, also eigenvalues of the sine$_\beta$ operator (Valko-Virag)