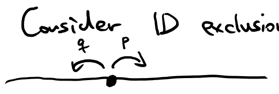
TASEP/ASEP

Monday, May 19, 2025 8:14 PM

Consider 1D exclusion, nearest neighbor jumps. (Asymmetric Simple Exclusion Process / ASEP)

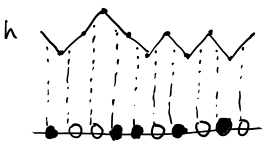


$x \to x+1$ rate p

$x \to x-1$ rate q

blocked if already occupied.

Can be encoded as random growth.



right jump: go up

left jump: go down.

Quantity of interest: at time t, how many particles to the right of 0?

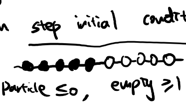
$\Leftrightarrow$ height of growth at 0 at time t.

(KPZ fixed point). For $h^{(1)}, h^{(2)}, \dots$ such that $x \mapsto -n^{-\frac{1}{3}} h_0^{(n)}(n^{\frac{2}{3}}x)$ converges to some $f: \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$
[$p = q+1$]

Then $x \mapsto -n^{-\frac{1}{3}}\left(h_n^{(n)}(n^{\frac{2}{3}}x) - \frac{n}{2}\right)$ converges to some random process $g: \mathbb{R} \to \mathbb{R}$, with distribution functions given by Fredholm determinants.

Konstantin-Quastel-Remenik, 17', for $p=1, q=0$ (Totally Assymetric Simple Exclusion Process / TASEP)

Today: explain "exact-solvability", mostly through TASEP with step initial condition, one point distribution.



Particle ≤ 0, empty ≥ 1

Transition Probability.

$G(x_1, \dots, x_n; t; y_1, \dots, y_n)$: Prob[Particles at $x_1 < x_2 < \dots < x_n$ at time t, starting from $y_1 < y_2 < \dots < y_n$ at time 0]

Time evolution of G (Kolmogorov forward equation) (normalize to $p+q=1$)

$n=1$: $\frac{d}{dt} G(x;t) = pG(x-1;t) + qG(x+1;t) - G(x;t)$; $G(x;0) = \delta_{x=y}$

$n=2$: $x_1 \le x_2 - 2$: $\frac{d}{dt} G(x_1, x_2; t) = pG(x_1-1, x_2; t) + qG(x_1+1, x_2; t) + pG(x_1, x_2-1; t) + qG(x_1, x_2+1; t) - 2G(x_1, x_2; t)$ (*)

$x_1 = x_2 - 1$: $\dots = pG(x_1-1, x_2; t) + qG(x_1, x_2+1; t) - G(x_1, x_2; t)$

$G(x_1, x_2; 0) = \delta_{x_1=y_1}\delta_{x_2=y_2}$

Can always use (*), plus boundary condition $pG(x,x;t) + qG(x+1,x+1;t) = G(x, x+1; t)$.

General n.

$$\frac{d}{dt} G(x_1, \dots, x_n; t) = \sum_{i=1}^{n} \left(pG(\dots, x_i - 1, \dots; t) + qG(\dots, x_i+1, \dots; t) - G(\dots, x_i, \dots; t)\right)$$

boundary condition: $pG(\dots, x, x, \dots; t) + qG(\dots, x+1, x+1, \dots; t) = G(\dots, x, x+1, \dots; t)$

initial condition: $G(x_1, \dots x_n; 0) = \delta_{x_1=y_1} \cdots \delta_{x_n=y_n}$

Task: solve this equation!

Idea: think of as $\frac{d}{dt} G = LG$; find eigenfunctions of the operator L! (Bethe ansatz)

For $L\Psi = \lambda \Psi$,

$n=1$: $p\Psi(x-1) + q\Psi(x+1) - \Psi(x) = \lambda\Psi(x)$; take $\Psi(x) = z^x$, then $\lambda = p/z + qz - 1$.

$n=2$: $p\Psi(x_1-1, x_2) + q\Psi(x_1+1, x_2) + p\Psi(x_1, x_2-1) + q\Psi(x_1, x_2+1) - 2\Psi(x_1, x_2) = \lambda\Psi(x_1, x_2)$

$p\Psi(x,x) + q\Psi(x+1, x+1) = \Psi(x, x+1)$

Take $\Psi(x_1, x_2) = A_{12} z_1^{x_1} z_2^{x_2} + A_{21} z_1^{x_2} z_2^{x_1}$; $\lambda = p/z_1 + qz_1 + p/z_2 + qz_2 - 2$,

$(A_{12} + A_{21})(p + qz_1z_2) = A_{12}z_2 + A_{21}z_1 \Rightarrow \frac{A_{21}}{A_{12}} = -\frac{p + qz_1z_2 - z_2}{p + qz_1z_2 - z_1}$

General n: $\Psi(x_1, \dots, x_n) = \sum_{\sigma \in S_n} A_\sigma \prod_{i=1}^{n} z_{\sigma_i}^{x_i}$ $\quad \lambda = \sum_{i=1}^{n} p/z_i + qz_i - 1$

where $A_\sigma = \operatorname{sgn}\sigma \prod_{i<j} \frac{p + qz_{\sigma(i)}z_{\sigma(j)} - z_{\sigma(i)}}{p + qz_iz_j - z_i}$

Then $G(x_1, \dots, x_n; t)$ with initial $\delta_{x_1=y_1} \cdots \delta_{x_n=y_n}$ can be written as linear combination of these eigenfunctions:

$$G(x_1, \dots x_n; t; y_1, \dots, y_n) = \sum_{\sigma \in S_n} \oint_c \cdots \oint_c dz_1 \cdots dz_n \, A_\sigma \prod_{i=1}^{n} z_{\sigma(i)}^{x_i - y_{\sigma(i)} - 1} \, e^{t \sum_{i=1}^{n} p/z_i + qz_i - 1}$$

Here c small circle enclose 0.

(check initial condition: take $t=0$, get $\delta_{x_1=y_1} \cdots \delta_{x_n=y_n}$)

Main difficulty: how to analyze it?

For TASEP ($p=1, q=0$) this is determinantal.

$G(x_1, \dots, x_n; t; y_1, \dots y_n) = \det\left(F_{k-j}(x_k - y_j; t)\right)_{j,k=1}^{n}$

Here $F_n(x;t) = \frac{1}{2\pi i} \oint_{0,1} \frac{dz}{z^{x+1}} (1 - \frac{1}{z})^{-n} e^{-(1-z)t}$

$\downarrow$ contour enclose 0 & 1.

This can be derived use Bethe Ansatz

Alternative, can directly check it satisfies Kolmogorov forward equation, using the facts

$F_{n+1}(x;t) = \sum_{y \ge x}^{\infty} F_n(y;t)$ (i)

$\frac{d}{dt} F_n(x;t) = F_n(x-1;t) - F_n(x;t)$ (ii)

Now suppose $y_i = -n + i$, find the probability of all n particles ≥ 1 at time t:

$$\Rightarrow \sum_{1 \le x_1 < \dots < x_n} \det\left(F_{k-j}(x_k + n - j; t)\right)_{j,k=1}^{n} \underset{\text{using (i) in columns}}{=} \sum_{1 \le x_1 < \dots < x_n} \prod_{j<k} (x_k - x_j) \cdot \det\left(F_{-j}(x_k + n - j; t)\right)_{j,k=1}^{n}$$

$$\underset{\text{using (i) in rows}}{=} \sum_{1 \le x_1 < \dots < x_n} \prod_{j<k} (x_k - x_j) \cdot \det\Big(\underbrace{F_0(x_k + n - j; t)}_{\frac{t^{x_k+n-j}}{(x_k+n-j)!} \cdot e^{-t}}\Big)_{j,k=1}^{n}$$

Further expand det, replace $x_i \to x_i + n - 1$

$$\Rightarrow \frac{1}{Z} \cdot \sum_{x_1, \dots x_n \ge n} \underbrace{\prod_{j<k} (x_k - x_j)^2 \prod_{j=1}^{n} \frac{t^{x_j}}{x_j!}}_{\text{discrete } \beta=2 \text{ ensemble with } V(x) = \frac{t^x}{x!}}.$$

$\underset{\text{Using (i) in columns}}{=} \det\left(F_{k-j+1}(n+k-j; t)\right)_{j,k=1}^{n} \underset{\text{using (ii)}}{=} \int_0^t dt_1 \cdots \int_0^t dt_n \det\left(F_{k-j}(n+k-j-1; t_k)\right)_{j,k=1}^{n}$

$$\underset{\text{using (i) for columns, (ii) for rows}}{=} \frac{1}{n!} \int_0^t dt_1 \cdots \int_0^t dt_n \prod_{j<k} (t_k - t_j) \det\Big(\underbrace{F_0(n-j; t_k)}_{\frac{t_k^{n-j}}{(n-j)!} e^{-t_k}}\Big)_{j,k=1}^{n}$$

$$\Rightarrow \frac{1}{\tilde{Z}} \int_{[0,t]^n} dt_1 \cdots dt_n \prod_{j<k} (t_k - t_j)^2 \prod_{j=1}^{n} e^{-t_j}$$

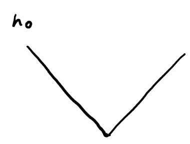
(Laguerre Unitary Ensemble in Wishart matrix)

$\mathbb{P}(X_1^{(n)} \le t)$

Can do e.g. steepest descent method, or orthogonal polynomials.

Note that this is the probability that $[\underbrace{\#\text{particles (to the right of 0 at time } t)}_{N(t)} \ge n]$

for initial being $\underbrace{\text{one particle at each } \mathbb{Z}_{\le 0}}_{\text{(Inifinte step initial)}}$. $\longrightarrow$ growth model:



$\Rightarrow \mathbb{P}\left[\frac{N(t) - t/4}{2^{-4/3} t^{1/3}} \ge -s\right] \to F_2(s)$ as $t \to \infty$

$\downarrow$

CDF of Tracy-Widom GUE

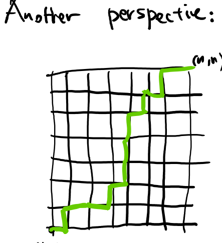
In terms of KPZ fixed point: if start with $f(0) = 0$, $f(x) = -\infty$ $\forall x \ne 0$

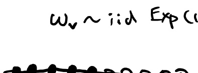
then $g(0) \sim$ Tracy-Widom GUE

More general initial: hard analysis to analyze the above determinant.
(for TASEP)

Another perspective: Last-Passage Percolation.

$w_v \sim$ iid Exp(1) for $v \in \mathbb{Z}^2$





$w_{(i,j)}$: waiting time for particle $\# i$ jump to hole $\# j$

$T_{(1,1),(n,n)} = \max_{\gamma} \sum_{v \in \gamma} w_v$ = First time n particles to the right of 0.

$\downarrow$

over all upright path from $(1,1)$ to (n,n)

proof. $T_{(1,1),(i,j)} = \max\{T_{(1,1),(i,j-1)}, T_{(1,1),(i-1,j)}\} + w$

$\downarrow$

Time when particle i jumps over hole j, by induction in i,j.

Therefore $n^{-1/3}(T_{(1,1),(n,n)} - 4n) \to$ Tracy-Widom GUE, as $n \to \infty$.

How about TASEP with general initial data?