MA 17: HOW TO SOLVE IT
HANDOUT 1: GENERAL PROBLEM-SOLVING STRATEGIES

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Try smaller cases: consider plugging in smaller numbers, and do examples.

Problem 1.
There are 99 baskets, each containing some apples and some bananas. Prove the following: no matter how many apples and bananas each basket contains, one can always choose 50 baskets, such that these 50 baskets (in total) contain at least half of all the apples and half of all the bananas.

Problem 2. (Putnam 2024, B1)
Let n and k be positive integers. The square in the ith row and jth column of an n-by-n grid contains the number $i + j - k$. For which n and k is it possible to select n squares from the grid, no two in the same row or column, such that the numbers contained in the selected squares are exactly $1, 2, \ldots, n$?

Problem 3.
For real numbers $a_1, a_2, \ldots, a_n$, the sum of any two (distinct) of them is non-negative. Let $x_1, x_2, \ldots, x_n$ be non-negative real numbers, satisfying $x_1 + x_2 + \cdots + x_n = 1$. Show that

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \geq a_1x_1^2 + a_2x_2^2 + \cdots + a_nx_n^2.$$

Induction principle (strong form)*: For a statement $P(n)$ that depends on an integer n, if*

(i) *for some integer a and non-negative integer m, all of $P(a), P(a+1), \ldots, P(a+m)$ are true,*
(ii) *for every integer $k > a + m$, $P(j)$ true for all $a \leq j < k$ implies $P(k)$ true.*

Then $P(n)$ is true for all $n \geq a$.

Problem 4.
In a system of 101 artificial satellites, each satellite monitors its nearest neighbor. Suppose that the distances between any two satellites are mutually distinct. Prove that there must exist at least one satellite that is not monitored by any other satellite.

Problem 5. (Putnam 2023, B3)
A sequence $y_1, y_2, \ldots, y_k$ of real numbers is called *zigzag* if $k = 1$, or if $y_2 - y_1, y_3 - y_2, \ldots, y_k - y_{k-1}$ are nonzero and alternate in sign. Let $X_1, X_2, \ldots, X_n$ be chosen independently from the uniform distribution on $[0, 1]$. Let $a(X_1, X_2, \ldots, X_n)$ be the largest value of k for which there exists an increasing sequence of integers $i_1, i_2, \ldots, i_k$ such that $X_{i_1}, X_{i_2}, \ldots, X_{i_k}$ is zigzag. Find the expected value of $a(X_1, X_2, \ldots, X_n)$ for $n \geq 2$.

Problem 6.
Let $A = \{a_1, a_2, \ldots, a_{100}\}$ be a set of 100 numbers, with $a_1 < a_2 < \cdots < a_{100}$. Suppose that

$\phi : A \to A$ is a bijection from A to itself, satisfying

$$a_1 + \phi(a_1) < a_2 + \phi(a_2) < \cdots < a_{100} + \phi(a_{100}).$$

Show that ϕ must be the identity map, i.e., $\phi(a_i) = a_i$ for each $i = 1, 2, \ldots, 100$.

Pigeonhole principle*: If n items are put into m containers, with $n > mk$, then at least one container must contain at least $k+1$ items. (Here m, n, k are positive integers.)*

Problem 7.
Arbitrarily choose $n+1$ different numbers from $1, 2, \ldots, 3n$. Prove that there are always two numbers (among these $n+1$ numbers), whose difference is $\geq n$ and $\leq 2n$.

Problem 8. (Putnam 2010, B3)
There are 2010 boxes labeled $B_1, B_2, \ldots, B_{2010}$, and $2010n$ balls have been distributed among them, for some positive integer n. You may redistribute the balls by a sequence of moves, each of which consists of choosing an i and moving *exactly* i balls from box B_i into any one other box. For which values of n is it possible to reach the distribution with exactly n balls in each box, regardless of the initial distribution of balls?

Proof by contradiction*: First, assume that the statement is false. Then, a sequence of logical deductions yields a conclusion that contradicts either the hypothesis, or a fact known to be true. This contradiction implies that the original statement must be true.*

Problem 9.
For a function $f : [0, 1] \to \mathbb{R}$, assume that $f(0) = f(1)$, and $|f(x) - f(y)| < y - x$ for any $0 \leq x < y \leq 1$. Prove that $|f(x) - f(y)| < \frac{1}{2}$ for any $0 \leq x < y \leq 1$.

Problem 10. (Putnam 2012, A1)
Let $d_1, d_2, \ldots, d_{12}$ be real numbers in the open interval $(1, 12)$. Show that there exist distinct indices i, j, k such that d_i, d_j, d_k are the side lengths of an acute triangle.

Extremal elements*.*

Problem 11.
100 soccer teams participate in a tournament, with one match between every two teams (resulting in each team playing 99 games). Suppose that no team wins all 99 games. Prove that there exist three teams A, B, and C, such that A beats B, B beats C, and C beats A.

Problem 12. (IMO 1988)
Let a and b be positive integers such that $ab+1$ divides a^2+b^2. Show that $\frac{a^2+b^2}{ab+1}$ is the square of an integer.

Reminder PSet 1 to be released tomorrow, and due by the end of Oct 8 (on Canvas).