MA 17: HOW TO SOLVE IT
HANDOUT 4: PROBABILITY AND EXPECTATION

Instructor Lingfu Zhang (lingfuz@caltech) **Office** Linde Hall 358
TA Minghao Pan (mpan2@caltech)

Probability.
Sample space: the set of all outcomes for a probability trial. Each outcome x has a certain probability.
An event S is a subset of the sample space.
For example, when randomly choose one card from a set of cards, the sample space is the set of all 54 cards, and each card has probability $\frac{1}{54}$. For the event of 'a King is chosen', its probability is $\frac{4}{54} = \frac{2}{27}$.
As another example, for X being uniformly random from the interval $[0, 1]$, then event $S = [0.3, 0.9]$ has probability 0.6.

Problem 1.
Shanille O'Keal shoots free throws on a basketball court. She hits the first and misses the second, and thereafter the probability that she hits the next shot is equal to the proportion of shots she has hit so far. What is the probability she hits exactly 50 of her first 100 shots?

Problem 2.
A stick is broken at two points chosen uniformly at random. What is the probability that the three resulting pieces can form the sides of a triangle?

Problem 3. (Putnam 2021, B1)
Suppose that the plane is tiled with an infinite checkerboard of unit squares. If another unit square is dropped on the plane at random with position and orientation independent of the checkerboard tiling, what is the probability that it does not cover any of the corners of the squares of the checkerboard?

Expectation.
For a random variable X whose value is a positive integer, its expectation is

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} \mathbb{P}[X = k]k.$$

Expectation is linear: for two random variables X and Y, $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$.

Problem 4.
Let π be a random permutation of the numbers $1, 2, \ldots, n$. What is the expected length of the cycle containing 1?

Problem 5. (Putnam 2022, A4)
Suppose that $X_1, X_2, \ldots$ are real numbers between 0 and 1 that are chosen independently and uniformly at random. Let $S = \sum_{i=1}^{k} X_i/2^i$, where k is the least positive integer such that $X_k < X_{k+1}$,

or $k = \infty$ if there is no such integer. Find the expected value of S.

For the remaining problems, try to find recurrence.

Problem 6. (Putnam 2001, A2)
You have coins $C_1, C_2, \ldots, C_n$. For each k, C_k is biased so that, when tossed, it has probability $1/(2k+1)$ of falling heads. If the n coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of n.

Problem 7. (Putnam 2024, B4)
Let n be a positive integer. Set $a_{n,0} = 1$. For $k \geq 0$, choose an integer $m_{n,k}$ uniformly at random from the set $\{1, \ldots, n\}$, and let

$$a_{n,k+1} = \begin{cases} a_{n,k} + 1, & \text{if } m_{n,k} > a_{n,k}; \\ a_{n,k}, & \text{if } m_{n,k} = a_{n,k}; \\ a_{n,k} - 1, & \text{if } m_{n,k} < a_{n,k}. \end{cases}$$

Let $E(n)$ be the expected value of $a_{n,n}$. Determine $\lim_{n\to\infty} E(n)/n$.

Problem 8. (Putnam 2020, A4)
Consider a horizontal strip of $N+2$ squares in which the first and the last square are black and the remaining N squares are all white. Choose a white square uniformly at random, choose one of its two neighbors with equal probability, and color this neighboring square black if it is not already black. Repeat this process until all the remaining white squares have only black neighbors. Let $w(N)$ be the expected number of white squares remaining. Find

$$\lim_{N\to\infty} \frac{w(N)}{N}.$$

Reminder PSet 3 due by the end of tomorrow. PSet 4 to be released tomorrow, and due by the end of Nov 5 (on Canvas). The third problem session (to discuss PSet 3) will be on Oct 30, 7-8 PM, at the same location.