MA 17: HOW TO SOLVE IT
MOCK PUTNAM 1

A1. A group of n spies, each initially possessing a unique piece of intelligence, can communicate via phone calls. In each call, two spies exchange all the information they currently know. Prove that when $n \geq 4$, it is possible for all n spies to learn all the intelligence using $2n-4$ phone calls.

A2. Let ABC be a triangle, and let points D, E, and F lie on the sides BC, CA, and AB, respectively. Consider the three triangles AEF, BDF, and CDE. Prove that the area of at least one of the triangles AEF, BDF, or CDE is less than or equal to $\frac{1}{4}$ of the area of triangle ABC.

A3. Finitely many positive integers are written on a chalk board. One can choose two of them, erase them, and replace them with their greatest common divisor and least common multiple. Prove that eventually the numbers on the board do not change.

For two positive integers, their greatest common divisor is the largest positive integer that divides both of them, and the least common multiple is the smallest positive integer that is divisible by both of them.

A4. Let $a_0 = 1$, and let $a_n = \ln(1 + a_{n-1})$ for $n \geq 1$. Determine whether $\sum_{n=1}^{\infty} a_n^2$ converges.

A5. A chess tournament took place between $2n+1$ players. Every player played every other player once, with no draws. In addition, each player had a numerical rating before the tournament began, with no two players having equal ratings. It turns out there were exactly k games in which the lower-rated player beat the higher-rated player. Prove that there is some player who won no less than $n - \sqrt{2k}$ and no more than $n + \sqrt{2k}$ games.

A6. In a finite sequence of real numbers, the sum of any 7 successive terms is negative and the sum of any 11 successive terms is positive. Determine the maximum number of terms in the sequence.