MA 17: HOW TO SOLVE IT
PROBLEM SET 2

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Due: 11:59PM of October 15. Late submissions will not be accepted.

Instructions:

- *Please submit solutions to at most three problems, written carefully. You are encouraged to work on more, but the TA will provide feedback on only three. I recommend selecting the three for which you most want feedback.*
- *Collaboration on homework before you have solved at least three problems independently is discouraged, though not strictly forbidden. You should make a serious effort to work through each problem independently before turning to classmates, the internet, or tools like ChatGPT.*
- *As long as your submission clearly shows attempts on at least three problems, you will earn 8 points toward the final grade. The correctness of your solutions will not affect your score, although feedback will be provided.*
- *Submission should be made through Canvas.*

Problem 1.
Prove that any finite set of people can be divided into two groups, so that each person knows at least as many people in the other group as in their own group. (Assume that knowing is mutual: A knows B if and only if B knows A.)

Problem 2.
Let a and b be positive integers. Show that if $4ab - 1$ divides $(a-b)^2$, then $a = b$.

Problem 3.
Solve in positive integers the equation

$$2^x \cdot 3^y = 1 + 5^z.$$

Problem 4.
Find the last two digits of 7^{7^7}.

Problem 5.
Show that for every prime p there is an integer n such that $2^n + 3^n + 6^n - 1$ is divisible by p.

Problem 6.
There are n points in the plane, satisfying the following property: for any two points, the straight line passing through them also passes through a third point (among these n points). Prove that all these n points are on the same line.

Hint: Assume that they are not on the same line. Consider all the point-to-line distances, and take the smallest one.

Problem 7. (Problem 8 in Handout 2)
Consider the integers $1, 2, \ldots, 2^k - 1$ for $k \geq 2$. Among them, some have an even sum of digits in their binary representation. Find the sum of all such integers.
(For example, the binary representation of 2 is 10, which has an odd sum of digits; while the binary representation of 3 is 11, which has an even sum of digits.)

Problem 8.
For each positive integer n, let $k(n)$ be the number of ones in the binary representation of $2023 \cdot n$. What is the minimum value of $k(n)$?

Reminder Problem session: Oct 9, 7PM to 8PM (to discuss PSet 1). Next Class: Oct 14, 7PM-10PM. (Mock Putnam 1; you will be free to leave early, and there will be snacks.)