MA 17: HOW TO SOLVE IT
MOCK PUTNAM 1 — FALL 2026
SOLUTIONS

A1. *Answer:* 580 *jumps.*

If a jump changes the coordinates by integers u, v, then $u^2 + v^2 = 25$. The possibilities are

$$(\pm 5, 0), \quad (0, \pm 5), \quad (\pm 3, \pm 4), \quad (\pm 4, \pm 3).$$

Thus each jump increases $x + y$ by at most 7. To reach $(2026, 2026)$ requires at least

$$\left\lceil \frac{4052}{7} \right\rceil = 579$$

jumps. But every possible jump changes $x + y$ by an odd integer. Since $x + y$ is even both at the beginning and at the end, the number of jumps must be even. The lower bound is therefore 580.

To attain it, take 288 jumps of each of $(3, 4)$ and $(4, 3)$, then two jumps of each of $(5, 0)$ and $(0, 5)$. These $576 + 4 = 580$ jumps have total displacement

$$288(3, 4) + 288(4, 3) + 2(5, 0) + 2(0, 5) = (2026, 2026).$$

A2. Number the positions from the top, starting at 1. Give a correctly placed card i a score of 2^{i-1}, and a misplaced card a score of 0. Let W be the total score. It is an integer satisfying

$$0 \le W \le 1 + 2 + \cdots + 2^{n-1} = 2^n - 1.$$

Suppose the top card is $k > 1$. Reversing the first k cards puts card k in position k, where it was not before, and therefore gains 2^{k-1} points. For each $i > k$, position i is untouched, so whether card i is correctly placed does not change. Some smaller cards may cease to be correctly placed, but their total possible loss is at most

$$1 + 2 + \cdots + 2^{k-2} = 2^{k-1} - 1.$$

Thus every operation increases W by at least 1.

A bounded integer cannot keep increasing in this way forever. Hence only finitely many operations occur. Since an operation is prescribed whenever the top card is not 1, the process must stop with card 1 on top. This also covers $n = 1$, when no operation is needed.

A3. First observe that, for distinct integers u, v, the integer $u - v$ divides $f(u) - f(v)$. Indeed, for each positive integer j,

$$u^j - v^j = (u - v) \sum_{t=0}^{j-1} u^{j-1-t} v^t,$$

and we can apply this identity to each term of f. The constant terms cancel, and the remaining coefficients are integers.

Let k be the smallest positive index with $a_k = 0$. If $k = 1$ or $k = 2$, the desired conclusion holds. Suppose for a contradiction that $k \ge 3$.

The values $a_0, a_1, \ldots, a_{k-1}$ are distinct. Otherwise, if $a_i = a_j$ with $0 \le i < j < k$, repeatedly applying f another $k - j$ times would give

$$a_{i+k-j} = a_k = 0.$$

Since $1 \le i + k - j < k$, this contradicts the choice of k.

Because $a_k = a_0$, the same k values repeat thereafter. For each integer i with $0 \le i \le k$, put

$$d_i = a_{i+1} - a_i.$$

These differences are nonzero, and $d_k = d_0$. For $0 \le i < k$, the divisibility observation gives

$$d_i \mid f(a_{i+1}) - f(a_i) = d_{i+1}.$$

Since all these differences are nonzero, divisibility implies

$$|d_0| \le |d_1| \le \cdots \le |d_k| = |d_0|.$$

All these absolute values equal some positive integer D.

Choose the largest of $a_0, \ldots, a_{k-1}$, and call it M. Regard the list cyclically, so that the predecessor of a_0 is a_{k-1}. The predecessor and successor of M are both smaller than M, and each differs from M by D. Both must therefore equal $M - D$.

Since $k \ge 3$, those are two different positions in the cycle. Their equal values contradict the distinctness proved above. Thus $k \le 2$, which proves $a_1 = 0$ or $a_2 = 0$.

B1. Relabel the missing integers so that $a_1 < \cdots < a_n$. We claim that $a_i \le 2i - 1$ for $1 \le i \le n$. Among the $a_i - 1$ positive integers smaller than a_i, exactly $i - 1$ are missing from S, so exactly $a_i - i$ belong to S.

If $r \in S$ and $1 \le r < a_i$, then $a_i - r$ cannot belong to S: otherwise closure under addition would imply $a_i = r + (a_i - r) \in S$, a contradiction. Also $1 \le a_i - r < a_i$, and different choices of r give different values of $a_i - r$.

We have therefore assigned a different missing integer below a_i to each member of S below a_i. Thus $a_i - i \le i - 1$, proving the claim. Summing it yields

$$a_1 + \cdots + a_n \le 1 + 3 + \cdots + (2n - 1) = n^2.$$

B2. *Answer:* 990 *keys.*

Every room must have keys held by at least 11 different guests. Otherwise, at least 90 of the 100 guests hold no key to that room. Choose 90 such guests as the returning group. They would have to occupy distinct rooms among the other 89 rooms, which is impossible by the pigeonhole principle.

Since a key opens just one room, this requires at least 11 keys per room, for a total of at least $90 \cdot 11 = 990$ keys. Issuing duplicate keys to the same guest cannot reduce this bound.

To attain it, choose 90 guests and give each one a single key, with these keys opening different rooms. Give each of the other 10 guests a full set of 90 keys. The number issued is $90 + 10 \cdot 90 = 990$.

If k of the returning guests have full sets, the other $90 - k$ guests take the distinct rooms corresponding to their individual keys. Exactly k rooms remain, and the guests with full sets can occupy them in any order. Thus every possible returning group can be accommodated.

B3. We prove the slightly stronger statement that the sum over good indices is nonnegative, and is positive whenever at least one good index exists. Use strong induction on the length of the sequence, with the empty sequence as the base case.

If there are no good indices, the sum is 0. Otherwise let i be the first good index, and choose the smallest $j \geq i$ for which

$$S = a_i + a_{i+1} + \cdots + a_j > 0.$$

For each k with $i \leq k \leq j$, minimality of j gives

$$a_i + \cdots + a_{k-1} \leq 0,$$

where the sum is empty when $k = i$. Consequently

$$a_k + \cdots + a_j = S - (a_i + \cdots + a_{k-1}) > 0.$$

Thus every index from i through j is good, and their terms have total $S > 0$.

There are no good indices before i. For an index after j, being good depends only on terms at that index or later, so its status is unchanged in the shorter sequence $a_{j+1}, \ldots, a_n$. By induction, the sum over the good indices in this tail is nonnegative. Adding the positive contribution S proves the claim.