Intro & spectral decomposition

Sunday, March 29, 2026 3:00 PM

Today: background / motivation from physics ; general spectral theory

General quantum physics:

Hilbert space $\mathcal{H}$, (i.e., a complex linear space, with inner product $\langle \cdot, \cdot \rangle$, such that
① $\langle x,y\rangle = \overline{\langle y,x\rangle}$ ② $\langle ax+by, z\rangle = a\langle x,z\rangle + b\langle y,z\rangle$, $a,b\in\mathbb{C}$
③ $\langle x,x\rangle > 0$ for $x\neq 0$ ④ complete with resp. to norm $\|x\| = \sqrt{\langle x,x\rangle}$)

(Any Hilbert space is complete with a countable orthonormal basis therefore isomorphic to $L^2(\mathbb{N})$)

Self-adjoint (bounded) operator H (i.e., $\langle x, Hy\rangle = \langle Hx, y\rangle$); the quantum Hamiltonian

evolution: $i\partial_t\psi = H\psi$, $\psi_t = e^{-itH}\psi_0$,

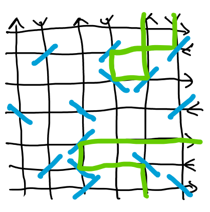
- Space $\mathcal{H}$ taken to be $L^2(X)$, for $X =$ e.g., $\{\pm 1\}^k$ (k-spin), $\mathbb{R}^d$, box $[-L,L]^d$, graph G, $\mathbb{Z}^d$, ...
  inner product $\langle f, g\rangle = \int_X f\bar{g}\, d\mu$
- One particle evolution: $H = -\Delta + V$ (Schrödinger operator)
  $\Delta\psi(x) = \begin{cases} \sum_{i=1}^d \partial_i^2\psi(x) & \text{continuous } X \\ \sum_{y\sim x} \psi(y) - \psi(x) & \text{discrete } X \end{cases}$
  $V: X\to\mathbb{R}$; $V\psi(x) = V(x)\psi(x)$ (multiplicative operator)
  ensure that H is self-adjoint.
  For normalized ψ_0, i.e. $\|\psi_0\|^2 = \langle\psi_0,\psi_0\rangle = 1$, $|\psi_t(x)|^2 =$ "Prob. particle at $x\in X$"
- Now consider disordered medium: e.g., one electron hopping in alloys, V is taken random.
  Philip W. Anderson (1958) investigated the tight-binding model:
  $\mathcal{H} = L^2(\mathbb{Z}^d)$, $H = -\Delta + V$. $V:\mathbb{Z}^d\to\mathbb{R}$ iid random. e.g., $V(x)\sim \mathrm{Unif}[-W,W]$ iid. (W: disorder strength, $W>0$)

★ Anderson's prediction: consider $\sum_{x\in\mathbb{Z}^d} |\psi_t(x)|^2\cdot|x|$, "expected distance to origin",
if $d=1,2$, or $d\geq 3$ and W is large: $\sup_{t\geq 0}\sum_{x\in\mathbb{Z}^d}|\psi_t(x)|^2\cdot|x| < \infty$ "localization" ; for electron: insulator
if $d\geq 3$, and W is small: $\sum_{x\in\mathbb{Z}^d}|\psi_t(x)|^2\cdot|x|\sim\sqrt{t}$, i.e., diffusive "delocalization" ; conductor,

Exercise: $H = -\Delta$ (i.e., $W=0$)
then $\sum_x |\psi_t(x)|^2|x| \sim t$
ballistic instead of diffusive
(idea: Fourier transform)

Compare to classical theory: particles do Brownian motion, always diffusive.
Many experiments support the presence of localization, and localization-delocalization transition (as disorder strength changes)
What is wrong with Brownian motion? Instead of "random walk on plain environment" should take "deterministic walk on random environment"

Consider another classical model
2d Manhattan pinball / random mirror model



alternating one-ways in $\mathbb{Z}^2$
with probability $p\in[0,1]$, put a police at each crossing
make a turn when seeing a police
Question: are all the trajectories finite? are there infinite ones?
When $p\geq\frac{1}{2}$, all trajectories finite
conjecture: for any $p>0$, all trajectories finite
(Linjun Li, 2020, for $p>\frac{1}{2}-\varepsilon$ for some $\varepsilon>0$)

Analog problems in $d\geq 3$: should $\exists$ a non-trivial phase transition.
(progress by Elboim, Gloria, Hernández, 2025, $d\geq 4$, typical trajectory length $\geq p^{-M}$)

Back to Anderson tight-binding: more from Spectral Theory.
If consider a finite system, e.g., $\mathcal{H} = L^2((\mathbb{Z}\cap[-L,L])^d)$, to get $\psi_t = e^{-itH}\psi_0$, can take all the eigenvalues/eigenfunctions of H
i.e. suppose all eigenvalues/eigenfunctions are $\underbrace{E_1, E_2,\cdots,E_N}_{\in\mathbb{R}}$, $\underbrace{\phi_1,\phi_2,\cdots,\phi_N}_{\text{orthonormal basis}}$ (since H self-adjoint) $(N\sim(2L)^d)$

$\Rightarrow \psi_t = \sum_{i=1}^N e^{-itE_i}\langle\psi_0,\phi_i\rangle\phi_i$

$\sum_x |\psi_t(x)|^2|x| = \sum_{x,i,j} e^{-it(E_i-E_j)}\langle\psi_0,\phi_i\rangle\overline{\langle\psi_0,\phi_j\rangle}\,\phi_i(x)\overline{\phi_j(x)}\cdot|x|$ $(*)$

suppose that each ϕ_i localized, i.e. $|\phi_i(x)|\lesssim e^{-c|x-x_i|}$ for some x_i (exponential localized)
$\sup_{t\geq 0}(*) \lesssim \sum_{i,j}\langle\psi_0,\phi_i\rangle\overline{\langle\psi_0,\phi_j\rangle}\cdot e^{-c|x_i-x_j|} \leq \sum_{i,j}\left(|\langle\psi_0,\phi_i\rangle|^2 + |\langle\psi_0,\phi_j\rangle|^2\right)e^{-c|x_i-x_j|} < C$ (independent of L)

if some ϕ_i not decay fast, $(*)$ grows in t ($\sup_{t\geq 0}(*)$ in terms of L), depending on ψ_0

- Example: $H = -\Delta$ (i.e., no potential)
  eigenfunction $\phi(x) = \frac{1}{\sqrt{N}}e^{2\pi i\lambda\cdot x}$, $\lambda\in((2L)^{-1}\mathbb{Z}\cap[0,1])^d$; not localized.
- Physical meaningful scale: L very large; consider $\mathcal{H} = L^2(\mathbb{Z}^d)$
  two types of "eigenfunctions": $\underbrace{\text{exponential decay}}_{\text{in } L^2(\mathbb{Z}^d)}$, or like $\underbrace{e^{2\pi i\lambda\cdot x}}_{\text{not in } L^2(\mathbb{Z}^d)}$, $\lambda\in[0,1]^d$; "generalized eigenfunction"

Need some functional analysis language.
For bounded operator H on a Hilbert space $\mathcal{H}$, (* Many statements below have unbounded operator version)
resolvent set: $\rho(H) = \{z\in\mathbb{C}: H - zI \text{ is bijective}\}$
spectrum: $\mathrm{Spec}(H) = \mathbb{C}\setminus\rho(H)$

Theorem $\mathrm{Spec}(H)$ is compact.
proof. ① $\rho(H)$ is open:
For $z\in\rho(H)$, $\|(H-zI)x\| > c\|x\|$ for some $c>0$, by open mapping theorem (as $H-zI$ is bounded surjective)
Then for any $w\in\mathbb{C}$, $|w-z|<c$, we can write
$H - wI = (H-zI)(I - \underbrace{(w-z)(H-zI)^{-1}}_{A})$, $\|A\|<1 \Rightarrow I-A$ invertible, with $(I-A)^{-1} = I + A + A^2 + \cdots$
$\Rightarrow H - wI$ invertible
② $\mathrm{Spec}(H)$ is bounded: when $z > \|H\|$, we have $(H-zI)^{-1} = -z(I + z^{-1}H + z^{-2}H^2 + \cdots)$

Theorem If H is self-adjoint, $\mathrm{Spec}(H)\subseteq\mathbb{R}$.
proof. For any $z\in\mathbb{C}\setminus\mathbb{R}$, $z = a+ib$, we can write
$\langle(H-zI)x, (H-zI)x\rangle = \langle(H-aI)x, (H-aI)x\rangle + \langle ibx, ibx\rangle + \underbrace{\langle(H-aI)x, ibx\rangle + \langle ibx, (H-aI)x\rangle}_{=0}$
$\Rightarrow \|(H-zI)x\| \geq |b|\cdot\|x\|$. In particular, $H - zI$ injective
Next: $\mathrm{Im}(H-zI)$ is dense.
Indeed, for any $x\in\mathcal{H}$, let $y\in\mathcal{H}$ be its projection on $\overline{\mathrm{Im}(H-zI)}$, i.e., $y\in\overline{\mathrm{Im}(H-zI)}$ with smallest $\|x-y\|$.
Then $\langle x-y, (H-zI)v\rangle = 0$ $\forall v\in\mathcal{H}$;
$\Rightarrow \langle(H-\bar{z}I)(x-y), v\rangle = 0 \Rightarrow (H-\bar{z}I)(x-y) = 0$.
but $H - \bar{z}I$ injective $\Rightarrow x = y$.
Then: $\mathrm{Im}(H-zI)$ is closed.
suppose $y_n = (H-zI)x_n\to y$, then $\|x_n - x_m\| \leq C\|y_n - y_m\|$, x_n is Cauchy-seq, so converges. For limit being x, $(H-zI)x = y$.

Example. For $-\Delta$ on $L^2(\mathbb{Z}^d)$, by Fourier transform, it is the operator of multiplying $2\sum_{i=1}^d(1-\cos\xi_i)$, on $L^2([0,2\pi]^d)$
$\Rightarrow -\Delta - zI$ is bijective iff $z\notin[0,4d] \Rightarrow \mathrm{Spec}(-\Delta) = [0,4d]$.

Example. For $H = -\Delta + V$, $V:\mathbb{Z}^d\to\mathbb{R}$ iid random, $V(0)\sim\mathbb{P}$ with compact support
(Kunz/Souillard, 1980) $\mathrm{Spec}(H) = [0,4d] + \mathrm{supp}(\mathbb{P})$, almost surely.

Weyl Criterion
For any self-adjoint operator H acting on Hilbert space $\mathcal{H}$,
$\mathrm{Spec}(H) = \{E\in\mathbb{R}: \exists x_n\in\mathcal{H}, \|x_n\|=1, \lim_{n\to\infty}\|(H-E)x_n\| = 0\}$.
proof. If $E\notin\mathrm{Spec}(H)$, $\|(H-E)x_n\| \geq \|(H-E)^{-1}\|^{-1}\cdot\|x_n\|$, not $\to 0$;
If $\inf_{\|x\|=1}\|(H-E)x\| > 0$, argue as above we have $\mathrm{Im}(H-E)$ is dense, since $\ker(H-E) = 0$ & $H-E$ self-adjoint,
$\underbrace{\text{also } \mathrm{Im}(H-E) \text{ is closed}}$
$\Downarrow$
$H - EI$ is surjective
also obviously $H - EI$ is injective
$\Rightarrow H - EI$ bijective.

Now we can prove (KS, 1980)
①. For $E\in\mathrm{Spec}(H) = \mathrm{Spec}(-\Delta+V)$, take $\psi_1,\psi_2,\cdots$ s.t. $\lim_{n\to\infty}\|(H-E)\psi_n\| = 0$, $\|\psi_n\| = 1$
$\Rightarrow \langle\psi_n, (H-E)\psi_n\rangle = \underbrace{\langle\psi_n, (-\Delta)\psi_n\rangle}_{\subseteq[0,4d]} + \underbrace{\langle\psi_n, V\psi_n\rangle}_{\subseteq\mathrm{supp}(\mathbb{P})} - E \to 0$
②. Take $\lambda\in[0,4d] = \mathrm{Spec}(-\Delta)$
$\mu\in\mathrm{Supp}(\mathbb{P})$
By Weyl: Take a sequence φ_n s.t. $\lim_{n\to\infty}\|(-\Delta-\lambda)\varphi_n\| = 0$; can truncate s.t. $\mathrm{supp}\,\varphi_n$ is finite.
Then a.s., there is $j_n\in\mathbb{Z}^d$ s.t. $|V(x+j_n) - \mu| < \frac{1}{n}$, $\forall x\in\mathrm{Supp}\,\varphi_n$.
(due to V being i.i.d.)
Take $\psi_n = \varphi_n(\cdot - j_n)$; then $\|(H-\lambda-\mu)\psi_n\| \leq \underbrace{\|(-\Delta-\lambda)\varphi_n\|}_{\to 0} + \underbrace{\|(V-\mu)\psi_n\|}_{\leq\frac{1}{n}} \to 0$ as $n\to\infty$
By Weyl: $\lambda+\mu\in\mathrm{Spec}(H)$.

Spectral Theorem & decomposition
Hilbert space $\mathcal{H}$, self-adj bounded operator H,
Idea: in finite dim, H is a Hermition matrix; can write $H\psi = \sum_{i=1}^N E_i\underbrace{\langle\psi,\phi_i\rangle\phi_i}_{\text{projection}}$
General Hilbert space $\mathcal{H}$: can write $H = \int_{\mathrm{Spec}(H)} E\, P_H(dE)$
P_H: "projection-valued measure"

Next: define P_H.
Consider $F_\psi: z\mapsto\langle\psi, (H-z)^{-1}\psi\rangle$, for $\psi\in\mathcal{H}$, $z\in\rho(H)$
Can prove: F_ψ is holomorphic in $\rho(H)$
(for any $z\in\rho(H)$, any w in a small nbhd of z, can write $(H-w)^{-1} = \sum_{k=0}^\infty (w-z)^k(H-z)^{-k-1}$)
Also, $\overline{F_\psi(z)} = F_\psi(\bar{z})$, F_ψ maps upper half-plane into itself
$(\overline{F_\psi(z)} = \langle(H-z)^{-1}\psi, \psi\rangle = \langle\psi, (H-\bar{z})^{-1}\psi\rangle = F_\psi(\bar{z}))$
$(F_\psi(z) - \overline{F_\psi(z)} = (z-\bar{z})\langle\psi, (H-z)^{-1}(H-\bar{z})^{-1}\psi\rangle = (z-\bar{z})\|(H-\bar{z})^{-1}\psi\|^2)$
(Here used that $(H-z_1)^{-1} - (H-z_2)^{-1} = (z_1-z_2)(H-z_1)^{-1}(H-z_2)^{-1}$)
$\Rightarrow F_\psi$ is a Herglotz-Pick function, or Nevanlinna function
Representation: $F_\psi(z) = \int\frac{d\mu_\psi(E)}{E-z}$, μ_ψ is the spectral measure of ψ; also define $\mu_{\psi,\phi}$ s.t. $\langle\psi, (H-z)^{-1}\phi\rangle = \int\frac{d\mu_{\psi,\phi}(E)}{E-z}$

(polarization: $\mu_{\psi,\phi} = \frac{1}{4}(\mu_{\psi+\phi} - \mu_{\psi-\phi} - i\mu_{\psi+i\phi} + i\mu_{\psi-i\phi})$)
$\mu_{\psi,\psi} = \mu_\psi$, $\mu_{\psi,\phi} = \overline{\mu_{\phi,\psi}}$, complex measure

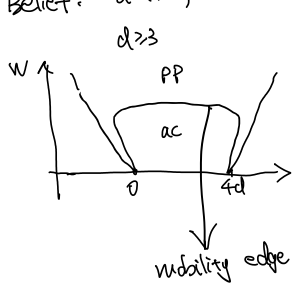
- For any $f\in L^\infty(\mathbb{R})$, can define operator $f(H)$ via $\langle\psi, f(H)\phi\rangle = \int f(E)\,d\mu_{\psi,\phi}(E)$ (Riesz representation) Check: $f(H)$ is self-adj.
- For any interval $I\subseteq\mathbb{R}$, define $P_H(I) = \mathbb{1}_I(H)$, which is a projection in $\mathcal{H}$ (i.e., $P_H(I)P_H(I) = P_H(I)$)
  $\Rightarrow H = \int_{\mathrm{Spec}(H)} E\, P_H(dE)$

$\to$ decomposition: $\mu_\psi = \mu_\psi^{pp} + \mu_\psi^{ac} + \mu_\psi^{sc}$ (decomposition of measures)
$\mathcal{H}^{pp}/\mathcal{H}^{ac}/\mathcal{H}^{sc}$: $\{\psi\in\mathcal{H}: \mu_\psi = \mu_\psi^{pp}/\mu_\psi^{ac}/\mu_\psi^{sc}\}$;

Example: $-\Delta$ on $L^2(\mathbb{Z}^d)$, it is the operator of multiplying $2\sum_{i=1}^d(1-\cos\xi_i)$, on $L^2([0,2\pi]^d)$
For any $\psi\in L^2([0,2\pi]^d)$, μ_ψ is the push-forward of the measure with density $|\psi(\xi)|^2$ on $[0,2\pi]^d$ under the map $\xi\mapsto 2\sum_{i=1}^d(1-\cos\xi_i)$, from $[0,2\pi]^d$ to $[0,4d]$.
$\Rightarrow \mu_\psi^{ac} = \mu_\psi$; $\mathcal{H}^{ac} = \mathcal{H}$,

- Orthogonal decomposition: $\mathcal{H} = \mathcal{H}^{pp}\oplus\mathcal{H}^{ac}\oplus\mathcal{H}^{sc}$ (for any ψ, can be written as the sum of $P_H(I_1)\psi + P_H(I_2)\psi + P_H(I_3)\psi$, where I_1, I_2, I_3 are supp of $\mu_\psi^{pp}, \mu_\psi^{ac}, \mu_\psi^{sc}$)
- Let $\mathrm{spec}^{pp}(H)$, $\mathrm{spec}^{ac}(H)$, $\mathrm{spec}^{sc}(H)$ be the spec of H restricted to these spaces,
  $\mathrm{spec}(H) = \mathrm{spec}^{pp}(H)\cup\mathrm{spec}^{ac}(H)\cup\mathrm{spec}^{sc}(H)$
- $\mathrm{spec}^{pp}(H) = \overline{\{E: E \text{ is eigenvalue of } H\}}$
  $\mathcal{H}^{pp} = \overline{\mathrm{span}\{\text{eigenfunctions of } H\}}$
  (for these, show that for any ψ s.t. μ_ψ is supported at one point, ψ must be an eigenfunction)

Now back to $H = -\Delta + V$, $V:\mathbb{Z}^d\to\mathbb{R}$, iid unif $[-W,W]$
$\mathrm{Spec}(H) = [-W, 4d+W]$ ($\mathrm{spec}^{pp}(H)$, $\mathrm{spec}^{ac}(H)$, $\mathrm{spec}^{sc}(H)$ are deterministic, Pastur's theorem, to be shown)
Belief: $d=1,2$, all p.p. Precise connection to the dynamics of e^{-itH} to be shown (RAGE theorem)
$d\geq 3$



Known: 1d ✓ (Carmona-Klein-Martinelli, 87)
$\geq 2d$: pp near edge, or W large enough
(2 methods, multi-scale analysis, and fractional moments; 1980s)
Frölich-Spencer, 83 Aizenman-Molchanov, 93

(singular potentials: Bourgain-Kenig 05
Ding-Smart 18
Li-Z. 19)

Better understood for some other graphs: trees, random graphs that are expanders, band matrices, Levy matrices, strips