Infinite regular tree

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Other than lattices, another natural family of graphs to consider is trees.

- For localization/delocalization, trees are expected to be easier, because the absense of loops makes easier to analyze the Green functions, via e.g., recursions.
- Phase diagrams can be different.

Setup: infinite ($d+1$)-regular tree, $d \geq 2$, denoted by $\mathbb{T}_d$

$H\psi(x) = -\sum_{y \sim x} \psi(y) + \lambda V(x)\psi(x)$; i.i.d. $\mathbb{T}_d \to \mathbb{R}$

$\downarrow$

Laplacian without diagonal terms; denoted by A

Again, since H is ergodic, $\mathrm{Spec}(H) = \mathrm{Spec}(A) + \lambda\,\mathrm{supp}(V)$.

- For recursion, we also consider $A \oplus V$ on $\mathbb{T}_d$, a rooted tree where each vertex has d children.

<u>Lemma</u> (Tree recursion). On any tree,

0. At any vertex x, $G(x,x;z) = \left(V(x) - z - \sum_{y \sim x} G_x(y,y;z)\right)^{-1}$

⊗ For any vertices x, y, connected by a path $x = x_1, x_2, \cdots, x_k = y$,

$G(x,y;z) = G(x,x;z) \prod_{i=2}^{k} G_{x_{i-1}}(x_i, x_i; z)$.

<u>proof</u>. For (1): consider the operator $V(x) \oplus 1_{\mathbb{T}_d}$, acting on $\mathbb{C} \oplus \ell^2(\mathbb{T}_d)$. Using that $A^{-1} = \vec{1}\vec{1}^T - A^{-1}(B-A)B^{-1}$, get $G(x,x;z) = \frac{1}{\sqrt{V(x)-z}} + \frac{1}{\sqrt{V(x)-z}} \sum_{y \sim x} G(x,y;z)$.

by ⊗, $G(x,y;z) = G(x,x;z)\, G_x(y,y;z) \Rightarrow G(x,x;z)\left(V(x) - z - \sum_{y \sim x} G_x(y,y;z)\right) = 1$.
(for $x \sim y$)

For ⊗: by geometric resolvent identity, for each i, $G(x,y;z) = G(x,x_i;z)\, G_{x_{i-1}}(x_i, x_i; z)$. Take product of $i = 2, \cdots, k$.

<u>Corollary</u>. For A on $\mathbb{T}_d$, the spectral measure at the root is

$\mu_0^A(dE) = \sqrt{4d - E^2}\, \frac{dE}{2\pi d}$ (semi-circle)

<u>proof</u>. By tree recursion, $G(x,x;z) = (-z - d\,G(x,y;z))^{-1} = \int \frac{\mu_0^A(dE)}{E - z}$.

$\Rightarrow d\,G(0,y;z)^2 + z\,G(0,y;z) + 1 = 0$

$G(x,y;z)^2 = \frac{z \pm \sqrt{z^2 - 4d}}{2d}$, take $z = E + i\eta$, and $\frac{1}{\pi} \lim_{\eta \downarrow 0} \mathrm{Im}\, G(0,0;E+i\eta)$.

Corollary: For A on $\mathbb{T}_d$, the spectral measure at any vertex x is

$\mu_x^A(dE) = \sqrt{4d - E^2}\, \frac{(d+1)\,dE}{2\pi(4d - E^2)}$ (Kesten-McKay law)

<u>proof</u>. By tree recursion, $G(x,x;z) = (-z - (d+1)\tilde{G}(0,0;z))^{-1}$; take $z = E + i\eta$, send $\eta \downarrow 0$, as above.
the Green function on $\tilde{\mathbb{T}}$.

In particular, $\mathrm{Spec}(A) = [-2\sqrt{d}, 2\sqrt{d}]$, a.c. spectrum.

- Lyapunov exponent: $L(z) := -\lim \frac{1}{x} G(0,x;z)$

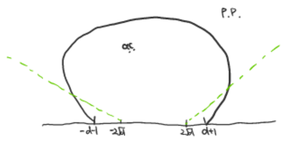
$|G(x,y;z)| = |G(x,x;z)|\, e^{-L(z)\,d(x,y)}$

$|\tilde{G}(0,x;z)| = |\tilde{G}(0,0;z)| \cdot e^{-L(z)\,d(x,y)}$

$L(E) = \log(d)$ when $|E| \leq 2\sqrt{d}$

$L(E) > \log(d)$ when $|E| > d+1$.

- <u>Specular</u> phase diagram:



Idea: need $\sum_x |G(0,x;z)|^s \begin{cases} < \infty & \text{localization} \\ = \infty & \text{delocalization} \end{cases}$ (Simon-Wolff)

Lyapunov exponent $L \approx$ decay rate of $|G(0,x;z)|$, in expected sense;
i.e., assuming i.i.d. roughness, by rank-1 perturbation, roughly,
$\mathbb{P}[|G(0,x;z)| \text{ of order } 1] \sim e^{-L(z)\,d(0,x)}$, independent for all x;
$\Rightarrow$ to get delocalization, need $\sum_x e^{-L(z)\,d(0,x)} < \infty$; $L(z) > \log(d)$.

Now let's work with $A \oplus V$, where $V: \mathbb{T}^d \to \mathbb{R}$ i.i.d., with bounded density P, s.t.

- Hölder: $P([x-\delta, x+\delta]) < C\delta^{\alpha}$ ($[x-\delta, x+\delta]$), $\forall x \in \mathbb{R}$, $\delta > 0$, for some $C > 0$.
- $\int |x|^{\alpha} P(x)\,dx < \infty$, for some $\alpha > 0$.

Can still define the Lyapunov exponent: $L_\lambda(z) := -\mathbb{E}[\log \tilde{G}(0,0;z)]$

Should instead consider moments; i.e., let $\varphi_\lambda(s;z) := \lim_{d(0,x) \to \infty} \frac{\log \mathbb{E}|G(0,x;z)|^s}{d(0,x)}$, for $s > -c$

<u>Theorem</u>: $\varphi_\lambda(s;z)$ is well-defined for all $s > -c$, $z \in \mathbb{C}_+$, convex and non-increasing in s; and for $s \in [0,1]$, $-s\,L_\lambda(z) \leq \varphi_\lambda(s;z) \leq -s \log\sqrt{d}$; $\lim_{s \downarrow 0} \varphi_\lambda(0;z) = -L_\lambda(z)$.

Besides, for $s < 1$, it can also be defined for $z \in \mathbb{R}$, for Lebesgue a.e. $E \in \mathbb{R}$, and $\lim_{\eta \downarrow 0} \varphi_\lambda(s; E+i\eta) = \varphi_\lambda(s;E)$; and the above also holds for $z = E$.

Moreover: the convergence is "uniform": for some constant $C_s(z)$,

$C_s(z)^{-1} e^{\varphi_\lambda(s;z)\,d(0,x)} \leq \mathbb{E}|G_\lambda(0,x;z)|^s \leq C_s(z)\, e^{\varphi_\lambda(s;z)\,d(0,x)}$.

This follows from recursion formulae, and sub/super-additivity arguments.
Here $C_s(z)$ is uniformly bounded, for z in compact set, $s \in [-c, 1-\varepsilon]$. [Here $s < 1$, for the same reason as in fractional moment method]

- Proof of the upper bound $\leq -\log\sqrt{d}$: suffices to prove $\varphi_\lambda(1;z) \leq -\log d$; by convexity.

Indeed: $\sum_x \mathbb{E}|G_\lambda(0,x;z)|^2 \leq \sum_x \mathbb{E}|G_\lambda(0,0;z)|^2 = \frac{\mathrm{Im}\, G_\lambda(0,0;z)}{\mathrm{Im}\, z} \leq \frac{1}{(\mathrm{Im}\, z)^2}$.

$= \langle (1-\vec{1}\vec{1}^T)\delta_0, (1-\vec{1}\vec{1}^T)\delta_0 \rangle = \langle \delta_0, (1-\vec{1}\vec{1}^T)^2 \delta_0 \rangle = (2-\vec{1}\vec{1}^T)\langle \delta_0, (1-\vec{1}\vec{1}^T)\delta_0 \rangle$

<u>Theorem</u>. If for $I \subseteq \mathbb{R}$ and Lebesgue a.e. $E \in I$, $\varphi_\lambda(1;E) < -\log d$, then $\mu_0^\omega(I) = 0$, for every x, almost surely.

Additionally assume: $\inf_{y \in \mathbb{R}} P(y) > 0$, for any $\delta > 0$. (i.e., non bounded distribution)

If for $I \subseteq \mathbb{R}$ and Lebesgue a.e. $E \in I$, $\varphi_\lambda(1;E) > -\log d$, then $\mu_0^\omega(I) > 0$, for every x, almost surely.

<u>proof</u> <u>localization side</u>: application of Simon-Wolff:
by properties of $\varphi_\lambda(s;z)$, for a.e. $E \in I$, we have $\sup_{\eta \in (0,\eta_0)} \varphi_\lambda(s; E+i\eta) < -\log d$, for some $s < 1$.

Then $\mathbb{E}\left[\left(\lim_{\eta \downarrow 0} \sum_y |G_\lambda(x,y;E+i\eta)|^s\right)^{\frac{s}{2}}\right]$

$\leq \liminf_{\eta \downarrow 0} \sum_y \underbrace{\mathbb{E}[|G_\lambda(x,y;E+i\eta)|^s]}_{\leq C_s(E+i\eta)\, e^{\varphi_\lambda(s;E+i\eta)\, d(x,y)}}$

$< \infty$

$\Rightarrow \mu_x^\omega(I) = 0$.

<u>delocalization side</u> Write $\sum_y |G(0,x;E)|^2 = |G(0,0;E)|^2 \cdot \underbrace{\sum_y \frac{|G(0,x;E)|^2}{|G(0,0;E)|^2}}_{|\tau(x,E)|^2}$

By rank-2 perturbation:

$\tau(x,E) = \frac{T(0,x;E)}{V(0) - G(0,0;E)}$

(Here $T(x,y;E) = T(y,x;E)$, and both $T(x,y;E)$ and $G(x,y;E)$ are independent of $V(x), V(y)$; in $\mathbb{T}_d$, $T(x,y;z) = G_{\{x,y\}}(x,y,z)$)

Idea: Let $t(0,x;E)$ be the quantile s.t. $\mathbb{P}[|T(0,x;E)| > t(0,x;E)] \geq \frac{3}{4}$.

Then with prob $\geq t(0,x;E)$, $|V(0) - G(0,x;E)| < 2|T(0,x;E)|$, by resampling $V(0)$.

Issue: $|G(0,x;E)|$ may be large, and its distribution can depend on $d(x,0)$.

Solution: consider $\Sigma(x;E)$, where $G(x,x;E) = (V(x) - \Sigma(x;E))^{-1}$. By linear algebra,

$\Sigma(x;E) - G(0,x;E) = \frac{T(x,0;E)^2}{V(0) - G(x,0;E)}$

Now consider the following events:

$T(x)$: $|T(x,x;E)|\,|z\,t(0,x;E)$. Then $\mathbb{P}[T(x)] \geq \frac{3}{4}$

$S(x)$: $|\Sigma(x;E)| \leq S$. S is chosen s.t. $\mathbb{P}[S(x)] \geq \frac{3}{4}$

$N(x)$: $|V(x) - G(x,0;E)| \geq |T(x,0;E)|$. $\mathbb{P}[N(x)] \geq \frac{3}{4}$, assuming $\mathbb{E}[1 \wedge |T(x,0;E)|] \to 0$ as $d(0,x) \to \infty$.

$T(x)$, $S(x)$, $N(x)$ are all independent of $V(x)$.

$\Rightarrow \mathbb{P}(|T(x,E)| \geq \frac{1}{2} \mid T(x) \cap S(x) \cap N(x)) \geq t(0,x;E)$

(Since under $T(x) \cap S(x) \cap N(x)$, $|\Sigma(x;E) - G(0,x;E)| \leq |T(x,0;E)|$; $V(x;E) \geq \frac{1}{2}$ as long as $|V(x) - \Sigma(x;E)| < t(0,x;E)$.)

$\Rightarrow \mathbb{E}\left[\sum_{x: d(0,x) \geq R} |T(x,E)|^2\right] \gtrsim \sum_{x: d(0,x) \geq R} t(0,x;E)$.

If $\sum_x t(0,x;E) = \infty$, $\sum_x \mathbb{E}|T(x;E)|^2 = \infty$. But we want a.s. $\sum_x |T(x;E)|^2 < \infty$.

- Second moment: Let $N_R :=$ # of x s.t. $T(x) \cap S(x) \cap N(x)$ holds, and $\underbrace{|V(x) - \Sigma(x;E)| < t(0,x;E)}_{\mathcal{E}_x}$.

Already have: $\mathbb{E}N_R \to \infty$ as $R \to \infty$

Next: $\frac{\mathbb{E}N_R^2}{(\mathbb{E}N_R)^2} < C$ as $R \to \infty$.

Note: $\mathbb{E}(N_R^2 - N_R) = \sum_{\substack{x \neq y \\ d(x,0) = d(y,0) \leq R}} \mathbb{P}[\mathcal{E}_x \cap \mathcal{E}_y]$

$\mathcal{E}_x \cap \mathcal{E}_y$: $G(x,x;E) \gtrsim (T(x,x;E))^{-1}$, $G(y,y;E) \gtrsim (t(0,y;E))^{-1}$

$\Leftrightarrow$ $U - \frac{Y}{W} < t(0,x;E)$, $W - \frac{Y}{U} < t(0,y;E)$

$Y: |T(x,y;E)|^2$, $U = V(y) - G(y,x;E)$, $W = V(x) - G(x,y;E)$

$\Rightarrow \mathbb{P}[\mathcal{E}_x \cap \mathcal{E}_y] \lesssim C\sqrt{t(0,x;E)\, t(0,y;E)}\, \mathbb{E}(1 \wedge (2\sqrt{t(0,x;E)\, t(0,y;E)} + 2|T(x,y;E)|))$

Sum over $x \neq y$, $d(0,x) = d(0,y) = R$, get

$\underbrace{\sum_{x,y,\, d(0,x)=d(0,y)=R} t(0,x;E)\, t(0,y;E)}_{\lesssim (\mathbb{E}N_R)^2} + \sum_{\substack{x,y \\ d(0,x)=d(0,y)=R}} t(0,x;E)\, \underbrace{\mathbb{E}(1 \wedge |T(x,y;E)|)}_{\text{Assume } \sum_{y: d(x,y)=R} \mathbb{E}[1 \wedge |T(x,y;E)|] \lesssim \sum_y t(0,y;E) \quad (*)}$

$\Rightarrow$ this also $\lesssim (\mathbb{E}N_R)^2$

- It now remains to lower bound $t(0,x;E)$ (i.e., sum over $x = \infty$)

upper bound $\mathbb{E}(1 \wedge |T(x,y;E)|)$ (i.e., $\to 0$ as $d(x,y) \to \infty$)

When $\varphi_\lambda(1;E) > -\log d$, (and $(*)$).

- can show $t(0,x;E) \gtrsim d^{(-1+\varepsilon)\, d(0,x)}$,
(since $T(0,x;E) = G_{\{0,x\}}(0,\hat{x};E)$, and use recursion + sub-additivity.)

$\Rightarrow \sum_x t(0,x;E) = \infty$

- Likewise, since $\varphi_\lambda(1;E) \leq -\log\sqrt{d}$, $\mathbb{E}(1 \wedge |T(x,y;E)|) \lesssim d^{-\frac{1}{2} d(x,y)} \to 0$ as $d(x,y) \to \infty$. Here ε is any number in $(0,1)$.

Also, for fixed x with $d(0,x) = R$

$\sum_{y: d(x,y) \geq R} \mathbb{E}(1 \wedge |T(x,y;E)|) \lesssim \sum_{j=1}^{2R} d^{j(\varepsilon-1)} < C_\varepsilon \frac{d^{R(\varepsilon-1)}}{d^{1-\varepsilon} - 1}$.

Since we already have $\sum_{x: d(0,x) \geq R} t(0,x;E) \gtrsim d^{\varepsilon R}$, for some $\varepsilon > 0$, by taking ε with $1 - \varepsilon < \frac{\varepsilon}{2}$, the conclusion follows.