RAGE theorem

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For dynamics, we consider $\langle \psi, e^{-itH}\phi\rangle$. (*)

Why? $e^{-itH}\phi$ is the wave at time t, starting from ϕ at time 0.

Take $\psi = \delta_x$ for some $x \in \mathbb{Z}^d$, $\Rightarrow |\langle \psi, e^{-itH}\phi\rangle|^2 =$ "$\mathbb{P}$[particle at x]"

From Spectral Theorem, we can write (*) as $\int e^{-itE}\, d\mu_{\psi,\phi}(E)$ ($F: x \mapsto e^{-itx}$ is in $L^\infty(\mathbb{R})$)

i.e. we get the Fourier transform of $\mu_{\psi,\phi}$!

Claim $\mu_{\psi,\phi}$ is abs. cont. w.r.t. μ_ϕ, and $h := \frac{d\mu_{\psi,\phi}}{d\mu_\phi} \in L^2(\mu_\phi)$, with $\int |h|^2\, d\mu_\phi \le \|\psi\|_2$

For this, we note that $|\mu_{\psi,\phi}(I)|^2 \le \mu_\psi(I)\,\mu_\phi(I)$ for each I.

[For any $z \in \mathbb{C}$, we have $\mu_{\psi+z\phi} = \mu_\psi + z\mu_{\psi,\phi} + \bar{z}\underbrace{\mu_{\phi,\psi}}_{\overline{\mu_{\psi,\phi}}} + |z|^2\mu_\phi \ge 0$

$\Rightarrow \begin{pmatrix} \mu_\psi & \mu_{\psi,\phi} \\ \mu_{\phi,\psi} & \mu_\phi \end{pmatrix}$ pos. semi-def]

uses

$\Rightarrow \mu_{\psi,\phi} \ll \mu_\phi$; Let h be the Radon-Nikodym derivative.

For any $\varphi \in L^2(\mu_\phi)$, $\left|\int \varphi h\, d\mu_\phi\right|^2 = \left|\int \varphi\, d\mu_{\psi,\phi}\right|^2 \le \left|\int |\varphi|^2 d\mu_\phi\right| \cdot \left|\int 1\, d\mu_\psi\right| = \|\varphi\|^2_{L^2(\mu_\phi)} \cdot \|\psi\|^2$

$\Rightarrow \varphi \mapsto \int \varphi h\, d\mu_\phi$ is a bounded functional $\underset{\text{Riesz representation}}{\Rightarrow} h \in L^2(\mu_\phi)$, $\int |h|^2 d\mu_\phi \le \|\psi\|_2$.

Now for (*), as $t \to \infty$, only pure point part of $\mu_{\psi,\phi}$ matters.

(Wiener) $\lim_{T\to\infty} \frac{1}{T}\int_0^T |(*)|^2\, dt = \sum_{E \in \operatorname{supp}(\mu^{pp}_{\psi,\phi})} |\mu^{pp}_{\psi,\phi}(\{E\})|^2 = \sum_{E \in \operatorname{supp}(\mu^{pp}_{\phi})} |\mu^{pp}_{\phi}(\{E\})|^2 \cdot |h(E)|^2$

proof. $= \frac{1}{T}\iiint_0^T e^{-itE+itE'}\, dt\, d\mu_{\psi,\phi}(E)\, d\overline{\mu_{\psi,\phi}(E')} \xrightarrow[T\to\infty]{} \iint \delta_{E,E'}\, d\mu_{\psi,\phi}(E)\, d\overline{\mu_{\psi,\phi}(E)} =$ RHS.

Can also bound the decay of (*) in time, if no pure point.

(Strichartz-Last) Suppose that μ_ϕ is α-Hölder, i.e., $\mu_\phi(I) < C|I|^\alpha$ for constant $C > 0$.

$\Rightarrow \frac{1}{T}\int_0^T |(*)|^2\, dt \le \frac{C_\phi \|\psi\|^2}{T^\alpha}$

proof $\le \frac{e}{T}\iiint e^{-\frac{t^2}{T^2}} |h(E)|\cdot|h(E')| \cdot e^{-itE+itE'}\, dt\, d\mu_\phi(E)\, d\mu_\phi(E')$

$\le e\sqrt{\pi} \iint |h(E)|^2 \cdot e^{-\frac{T^2}{4}|E-E'|^2}\, d\mu_\phi(E)\, d\mu_\phi(E')$

integrate E' give $\frac{C_\phi}{T^\alpha}$; note $\int |h(E)|^2\, d\mu_\phi(E) \le \|\psi\|$.

Ruelle-Amrein-Georgescu-Enss

For self-adj H in $\mathcal{H}$,

Then $\mathcal{H}^c = \{\psi \in \mathcal{H} \mid \lim_{n\to\infty} \lim_{T\to\infty} \frac{1}{T}\int_0^T \|P_n e^{-itH}\psi\|^2 dt = 0\}$ (1)

$(= \mathcal{H}^{ac} \oplus \mathcal{H}^{sc})$

$\mathcal{H}^{pp} = \{\psi \in \mathcal{H} \mid \lim_{n\to\infty} \sup_{t\in\mathbb{R}} \|(I - P_n) e^{-itH}\psi\| = 0\}$. (2)

Here $P_1, P_2, \dots$ is a seq. of compact operators, s.t. $\|(P_n - I)\psi\| \to 0$ for any $\psi \in \mathcal{H}$. (strong converge of operators)

can be approximated by finite rank operators.

proof. If $\psi \in \mathcal{H}^c$, by Wiener, $\frac{1}{T}\int_0^T \|P e^{-itH}\psi\|^2 dt \to 0$ for each compact operator P. (by first consider P = finite rank, then approximate)

If $\psi \in \mathcal{H}^{pp}$, write $\psi = \sum_{i=1}^\infty \psi_i$, each is eigenfunction (not normalized)

$\Rightarrow$ each ψ_i has $\sup_{t\in\mathbb{R}} \|(I - P_n) e^{-itH}\psi_i\| = \|(I - P_n)\psi_i\| \to 0$ as $n \to \infty$; then so is the sum (using $\sup_n \|P_n\| < \infty$)

Now for any $\psi \in \mathcal{H}$, write $\psi = \psi^c + \psi^{pp}$

the conditions in (1) and (2) contradict each other $\Rightarrow$ ψ^c has (1), not (2); ψ^{pp} has (2), not (1) $\Rightarrow$ if ψ has (1), must $\psi^{pp} = 0$; if ψ has (2), must $\psi^c = 0$.