Spectral calculus

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To show that $P_H(I)$ define a measure, need $P_H^2(I) = P_H(I)$, and $P_H(I)$ is self-adj.

Self-adj: $\langle \psi, P_H(I)\phi\rangle = \mu_{\psi,\phi}(I) = \overline{\mu_{\phi,\psi}(I)} = \overline{\langle \phi, P_H(I)\psi\rangle} = \langle P_H(I)\psi, \phi\rangle$.

$P_H^2(I) = P_H(I)$: Core step: for any $z, w \in \rho(H)$, we have

$$\langle \psi, (H-z)^{-1}(H-w)^{-1}\phi\rangle = \frac{1}{z-w}\left(\langle \psi, (H-z)^{-1}\phi\rangle - \langle \psi, (H-w)^{-1}\phi\rangle\right) = \frac{1}{z-w}\int \left((E-z)^{-1} - (E-w)^{-1}\right) d\mu_{\psi,\phi}$$
$$= \int (E-z)^{-1}(E-w)^{-1}\, d\mu_{\psi,\phi}$$

$$\Rightarrow d\mu_{\psi,(H-w)^{-1}\phi} = (E-w)^{-1} d\mu_{\psi,\phi}$$

$$\Rightarrow \langle P(I)\psi, (H-w)^{-1}\phi\rangle = \int_I (E-w)^{-1} d\mu_{\psi,\phi}$$

$$= \int (E-w)^{-1} d\mu_{P(I)\psi,\phi}$$

$$\Rightarrow d\mu_{P(I)\psi,\phi} = \mathbb{1}_I\, d\mu_{\psi,\phi}$$

$$\Rightarrow \mu_{P(I)\psi,\phi}(J) = \mu_{\psi,\phi}(I\cap J) \Rightarrow \langle P(I)\psi, P(J)\phi\rangle = \langle \psi, P(I\cap J)\phi\rangle$$

($= \langle \psi, P(I)P(J)\phi\rangle$ by conjugacy)

$$\Rightarrow P(I)P(J) = P(I\cap J)$$

<u>Implication</u>: $I\cap J = \phi \Rightarrow P(I)P(J) = 0$

$P(I)^2 = P(I)$

More generally, for any $f, g \in L^\infty(\mathbb{R})$, $(fg)(H) = f(H)\,g(H)$ (as self-adj operators)

(extend from $f = 1_I$, $g = 1_J$)