Ergodic operators: Pastur's theorem and DOS

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We've seen that for $H = -\Delta + V$ acting on $L^2(\mathbb{Z}^d)$, where $V \sim$ iid $\mathbb{P}$, with compact supp, almost surely $\mathrm{Spec}(H) = [0, 4d] + \mathrm{supp}(\mathbb{P})$

One can actually show: $\mathrm{Spec}^{pp}(H)$, $\mathrm{Spec}^{ac}(H)$, $\mathrm{Spec}^{sc}(H)$ are all deterministic.

proof. Let $T_x H = -\Delta + T_x V$, where $T_x V(y) = V(x+y)$. for some $x \in \mathbb{Z}^d$.

Then $\mathrm{Spec}^{\#}(T_x H) = \mathrm{Spec}^{\#}(H)$, almost surely, since $T_x H = T_x \circ H \circ T_{-x}$.
($\# = pp, ac, sc$)

$\Rightarrow$ For any open interval I, the event $I \cap \mathrm{Spec}^{\#}(H) \neq \phi$ (on the space of $(\mathbb{R}^{\mathbb{Z}^d}, \mathbb{P}^{\mathbb{Z}^d})$) is invariant under the operations $(T_x)_{x \in \mathbb{Z}^d}$. Therefore its probability $\in \{0, 1\}$

$\Rightarrow$ By take $I = (p, q)$ for all $p, q \in \mathbb{Q}$, $p < q$, we get that $\mathrm{Spec}^{\#}(H)$ is deterministic.

- More generally, this holds for all <u>ergodic operator</u>

Let (Ω, μ) be a probability space, and $(T_x)_{x \in I}$ an ergodic family of measure-preserving transformations.
(i.e., for any event $\mathcal{E}$ s.t. $T_x \mathcal{E} = \mathcal{E}$, for all $x \in I$, we must have $\mu(\mathcal{E}) \in \{0,1\}$.)

A random operator H defined over this prob space (acting on Hilbert space $\mathcal{H}$) is an <u>ergodic operator</u>, if for every $x \in I$, $w \in \Omega$, we have $H(T_x w)$ is unitarily equivalent to $H(w)$.

[$H(T_x w) = U^{\dagger} H(w) U$ for some unitary operator U]

(Pastur) Any ergodic operator has deterministic $\mathrm{Spec}^{\#}$ ($\# = pp, ac, sc$)

proof. Same as above, in the more abstract language. Use that $\mathrm{Spec}^{\#}(H) = \mathrm{Spec}^{\#}(U^{\dagger} H U)$ for unitary U.

- Another (less obvious) example.

almost Mathieu operator $H^{\lambda,\alpha}$ acting $L^2(\mathbb{Z})$, via $H^{\lambda,\alpha} f(x) = f(x+1) + f(x-1) + 2\lambda \cos(2\pi(u + \alpha x)) f(x)$

"randomness": $u \sim \mathrm{unif}[0,1]$

- α rational: periodic, equiv to finite dim problem.
- α irrational: ergodic operator. $\Rightarrow$ spec & decomposition deterministic

(Ten Martini Problem: for all $\lambda \neq 0$, irrational α, and all u, the spectrum is a Cantor set)

(Asked by Mark Kac, who offered ten Martini for solving it; also Barry Simon)

(Avila - Jitomirskaya, 05')

$\forall \alpha$ irrational:

- $\lambda > 1$: $\mathrm{Spec}(H^{\lambda,\alpha})$ is only pure point (Jitomirskaya, 99')
- $\lambda = 1$: $\mathrm{Spec}(H^{\lambda,\alpha})$ is only singular continuous (Gordon, Jitomirskaya, Last, Simon, 97')
- $0 < \lambda < 1$: $\mathrm{Spec}(H^{\lambda,\alpha})$ is only absolute continuous (Avila, 08')

(Hofstadter's butterfly, as α varies)

- Density of state (DOS) measure.

Now consider any ergodic operator H acting on $L^2(\mathbb{Z}^d)$, under the translations $(T_x)_{x \in \mathbb{Z}^d}$, with $H(T_x w) = U^{\dagger} H(w) U$ for $U := T_{-x}$. (Standard ergodic operators on $L^2(\mathbb{Z}^d)$)
(T_x are ergodic and measure preserving)

$\nu(I) := \mathbb{E}(\langle \delta_x, P_H(I) \delta_x \rangle) = \mathbb{E}\, \mu_{\delta_x, \delta_x}(I)$; this is independent of the choice of $x \in \mathbb{Z}^d$, since $H(w) \overset{d}{=} H(T_x w)$

Theorem For any $f \in C_0(\mathbb{R})$ (i.e., continuous with compact supp), $\underbrace{\frac{1}{|Q_L|} \sum_{x \in Q_L} \langle \delta_x, f(H) \delta_x \rangle}_{\text{trace of } f(H) \text{ in } Q_L}$ $\xrightarrow[L \to \infty]{} \int f \, d\nu$, where Q_L is the box $[-L, L]^d \cap \mathbb{Z}^d$, almost surely.

($\approx$ think of as $\sum_i f(\lambda_i)$)

(think of as limit of $\sum_i \delta_{\lambda_i}$)

(In another word, almost surely, $\frac{1}{|Q_L|} \sum_{x \in Q_L} \mu_{\delta_x, \delta_x} \to \nu$ weakly)

proof. $\mathrm{LHS} = \frac{1}{|Q_L|} \sum_{x \in Q_L} \langle \delta_0, f(H(T_x w)) \delta_0 \rangle$

Convergence by Birkhoff's theorem. ◻

- Note: DOS is ac does not mean ac spectrum!

e.g. consider multiplicative operator V, for $V \sim$ iid on $\mathbb{Z}^d$.

DOS can also be approximated by finite-dim operators.

Theorem Let $H_L(w) = H_L$ act on $L^2(Q_L)$, such that $\frac{1}{|Q_L|} \cdot \mathrm{tr}\left| H_L(w) - \mathbb{1}_{Q_L} H(w) \right| < \epsilon(L)$, where $\epsilon(L) \to 0$ as $L \to \infty$.

Then $\frac{1}{|Q_L|} \cdot \sum_{x \in Q_L} \langle \delta_x, f(H_L) \delta_x \rangle \xrightarrow[L \to \infty]{} \int f \, d\nu$, for any $f \in C_0(\mathbb{R})$.

proof. Suffices to prove this for $f : x \mapsto \frac{1}{x - z}$, $z \in \mathbb{C} \setminus \mathbb{R}$; use that $\mathrm{tr}\left(\frac{1}{A-z} - \frac{1}{B-z}\right) = \mathrm{tr}\left((A-z)^{-1}(A-B)(B-z)^{-1}\right) \le |\mathrm{tr}(A-B)| \cdot \underbrace{\|(A-z)^{-1}\| \cdot \|(B-z)^{-1}\|}_{\le \mathrm{Im}(z)^{-2}}$.

Why useful? Can approximate H by different boundary conditions: free, periodic, etc. in large box Q_L.