Green functions and perturbative theory

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To prove localization/delocalization, or pp/conti. spectrum of a random operator on $L^2(\mathbb{Z}^d)$

a very useful tool is Green function, i.e., $G(x,y;z) = \langle \delta_x, (H-z)^{-1}\delta_y \rangle$, for $x,y \in \mathbb{Z}^d$, $z \in \rho(H)$.

- Closely related to eigenfunction.
- Decay of $G(x,y;z)$ in $|x-y|$, in some sense, is equiv to localization.

<u>(Simon-Wolff criterion)</u>

$H = T + V$ be an operator on $L^2(\underset{\text{countable graph}}{G})$, where T is self-adj, deterministic, an $V: G \to \mathbb{R}$ random,

s.t. the law of $V(x) \mid \{V(y)\}_{y \neq x}$ is absolute continuous, for each $x \in V$. Let $I \subseteq \mathbb{R}$ be a Borel set.

- If for a.e. $E \in I$, almost surely
$$\lim_{\eta \downarrow 0} \sum_y |G(x,y;E+i\eta)|^2 < \infty,$$
then almost surely, the spectral measure μ_{δ_x} in I has no continuous part.
- If for a.e. $E \in I$, almost surely
$$\lim_{\eta \downarrow 0} \sum_y |G(x,y;E+i\eta)|^2 = \infty$$
then almost surely, the spectral measure μ_{δ_x} in I has no pure point part.

- To understand this, note that we can write $\sum_y |G(x,y;z)|^2 = \|(H-z)^{-1}\delta_x\|^2 = \int \frac{d\mu_{\delta_x}(u)}{|u-z|^2}$; thus
$\lim_{\eta \downarrow 0} \sum_y |G(x,y;E+i\eta)|^2 = \int \frac{d\mu_{\delta_x}(u)}{|E-u|^2} =: \gamma$, which is $< \infty$ or $= \infty$.

Now we <u>perturb/resample $V(x)$;</u> (fixed $\{V(y)\}_{y \neq x}$)
if $\gamma < \infty$, E is eigenvalue for some choice of $V(x)$; every other choice of $V(x)$, "no spectrum" at E, $\Rightarrow$ no cont. spec.
if $\gamma = \infty$, E is not eigenvalue for any choice of $V(x)$. $\Rightarrow$ no p.p. spec

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Next, we go back to linear algebra, and study perturbation of a finite dim matrix.

- H_0: finite, Hermitian matrix (equivalent to a self-adj operator in an L^2-space)

$H_v = H_0 + v|\delta_x\rangle\langle\delta_x|$, or $H_0 + v \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. with $\delta_x = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$\mathcal{H}_{H_v,x} = \mathrm{span}(\delta_x, H_v\delta_x, H_v^2\delta_x, \cdots) = \mathrm{span}((H_v - z)^{-1}\delta_x, z \in \mathbb{C}\setminus\mathbb{R})$

<u>Theorem.</u> The eigenvalues of H_v in $\mathcal{H}_{H_v,x}$ are precisely those E, s.t. $(H_0 - E)^{-1}_{xx} = G_0(x,x;E) = -v^{-1}$ (for $v \neq 0$)
with corresponding eigenvectors $\phi_{E,x} = (H_0 - E)^{-1}\delta_x$.

<u>proof.</u> $\frac{1}{H_v - z} - \frac{1}{H_0 - z} = -\frac{1}{H_v - z}\underbrace{(H_v - H_0)}_{v \cdot |\delta_x\rangle\langle\delta_x|}\frac{1}{H_0 - z}$

Proj on to δ_x

Apply both sides to δ_x

$\Rightarrow (H_v - z)^{-1}\delta_x = (H_0 - z)^{-1}\delta_x - v \cdot (H_v - z)^{-1}\delta_x \cdot \langle\delta_x, (H_0 - z)^{-1}\delta_x\rangle$.

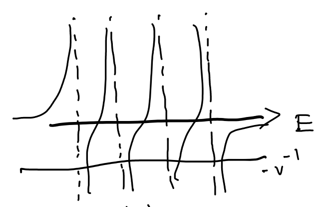
$\Rightarrow (1 + v\langle\delta_x, (H_0 - z)^{-1}\delta_x\rangle) \cdot (H_v - z)^{-1}\delta_x = (H_0 - z)^{-1}\delta_x$ (**)

$\Rightarrow (Hv\langle\delta_x, (H_0 - z)^{-1}\delta_x\rangle)\delta_x = (H_v - z)\underline{(H_0 - z)^{-1}\delta_x}$ (Also, if E is eigenvalue of H_v in $\mathcal{H}_{H_v,x}$, must have $\langle\delta_x, (H_v - z)^{-1}\delta_x\rangle \to \infty$ as $z \to E$)

as $z \to E$, the conclusion follows.

- Moreover, $\mu_{\delta_x}(\{E\})$ for H_v, equals $\frac{|\langle\phi_{E,x}, \delta_x\rangle|^2}{\langle\phi_{E,x}, \phi_{E,x}\rangle} = \frac{|\langle\delta_x, (H_0 - E)^{-1}\delta_x\rangle|^2}{\langle\delta_x, (H_0 - E)^{-2}\delta_x\rangle} \to$ corresponds to γ above.

Before taking the infinite version, note that
$(H_0 - E)^{-1}_{xx} = G_0(y,x;E) = \langle\delta_x, (H_0 - E)^{-1}\delta_x\rangle = \sum_{i=1}^n \frac{P_i}{E_i - E}$, where μ_{δ_x} for H_0 is $\sum_{i=1}^n P_i \delta_{E_i}$.



as v goes from $-\infty$ to ∞, the ordered statistics increase

Now <u>infinite (countable) version:</u>

H_0 act on $L^2(G)$, for G countable, self-adj; $H_v = H_0 + v|\delta_x\rangle\langle\delta_x|$

<u>Theorem</u> for any $v \neq 0$, $E \in \mathbb{R}$, the following two are equivalent:

① For H_v, $\mu_{\delta_x}(\{E\}) > 0$

② For $\gamma_0 := \int \frac{d\mu_{\delta_x}(u)}{|u-E|^2}$ of H_0, we have $\gamma_0 < \infty$, and $\lim_{\eta \downarrow 0}\langle\delta_x, (H_0 - E - i\eta)^{-1}\delta_x\rangle = \lim_{\eta \downarrow 0} G(x,x;E+i\eta) = -v^{-1}$

Moreover: if the above true, $(H_0 - E)^{-1}\delta_x$ would be the eigenfunction, and for H_v, $\mu_{\delta_x}(\{E\}) = \frac{1}{v^2\gamma_0}$.

<u>proof</u> Essentially same as finite dim.

(For ② $\Rightarrow$ ①, need $\gamma_0 < \infty$ so that $(H_0 - E)^{-1}\delta_x \in L^2$)

We can now proceed to prove <u>Simon-Wolff</u>

<u>implies</u>

For H_v with $v \neq 0$, $\mu^{pp}_{\delta_x}(\{E: \gamma_0(E) = \infty\}) = 0$ ⓐ

$\mu^c_{\delta_x}(\{E: \gamma_0(E) < \infty\}) = 0$ ⓑ

ⓐ obviously from ① $\to$ ② above.

<u>proof of ⓑ:</u> for each $v \neq 0$, and $E \in \mathbb{R}$ with $\gamma_0(E) < \infty$, if E is not an eigenvalue of H_v, we must have $\lim_{\eta \downarrow 0} G(x,x;E+i\eta) \neq -v^{-1}$.
(by ② $\to$ ① above)

Also, $\gamma_0(E) < \infty$ implies that $\lim_{\eta \downarrow 0} G_0(x,x;E+i\eta)$ is real

from (**) above, we get $\lim_{\eta \downarrow 0} G_v(x,x;E+i\eta)$ is real. $\Rightarrow$ $\mu^c_{\delta_x}$ for H_v on $\{E: \gamma_0(E) < \infty\}$ is zero.

This <u>almost</u> enough for <u>Simon-Wolff</u>

Do a resampling of $V(x)$ to v; we know that for H_v, $\mu^{pp}_{\delta_x}$ is supported on $\{E: \gamma_0(E) < \infty\}$, by ⓐ above.

However, $|I \cap \{E: \gamma_0(E) < \infty\}| = 0$, i.e. Lebesgue measure zero, does <u>not</u> imply $\mu^{pp}_{\delta_x}(I) = 0$!

Need one more step: <u>spectral averaging</u>.