Fractional moment

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We now aim to bound green function $G(x,y;z) = \langle \delta_x, (H-z)^{-1}\delta_y \rangle$, where $H = T+V$, for T self-adj, $V: \mathbb{G} \to \mathbb{R}$ potential, acting on $L^2(\mathbb{G})$.

(For random Schördinger, take $T := -\Delta - 2d$)

Random Walk expansion $$G(x,y;z) = \sum_{\gamma: x \to y} (-1)^k \cdot T(\gamma(0),\gamma(1))\, T(\gamma(1),\gamma(2)) \cdots T(\gamma(k-1),\gamma(k)) \cdot \prod_{i=0}^{k} \frac{1}{V(\gamma(i)) - z}.$$

(k = length of γ)

for any z with $\operatorname{Im} z > \|T\|$.

proof. $$(T+V-z)^{-1} = (V-z)^{-1} - (V-z)^{-1} T (T+V-z)^{-1}$$
$$= (V-z)^{-1} - (V-z)^{-1} T (V-z)^{-1} + (V-z)^{-1} T (V-z)^{-1} T (T+V-z)^{-1}$$
$$= \cdots$$
$$= \sum_{i=1}^{k-1} (V-z)^{-1} \left(T(V-z)^{-1}\right)^i - (V-z)^{-1} \underbrace{\left(T(V-z)^{-1}\right)^{k-1}}_{} \cdot T (T+V-z)^{-1}.$$

$$\|T(V-z)^{-1}\| \le \frac{\|T\|}{\operatorname{Im} z} < 1$$

Self-avoiding walk expansion $$G(x,y;z) = \sum_{\substack{\hat\gamma: x \to y \\ s.a.}} (-1)^k\, T(\hat\gamma(0),\hat\gamma(1)) \cdots T(\hat\gamma(k-1),\hat\gamma(k)) \cdot \prod_{i=0}^{k} \langle \delta_{\hat\gamma(i)}, \frac{1}{H_i - z} \delta_{\hat\gamma(i)} \rangle$$

H_i: restriction of H to $L^2(\mathbb{G} \setminus \{\hat\gamma(j)\}_{j<i})$

for any z with $\operatorname{Im} z > \|T\|$, or $\operatorname{Im} z > 0$ if $\mathbb{G}$ finite.

proof. Loop erase: using R-W expansion to sum over loops.

Theorem. For $H = T + \lambda V$ acting on $L^2(\mathbb{G})$, where:

- $\mathbb{G}$ is a finite, regular graph
- $|T(x,y)| = \begin{cases} 1 & x \sim y \\ 0 & \text{else} \end{cases}$
- $V: \mathbb{G} \to \mathbb{R}$ iid, with each $V(x)$ prob density $\rho \in L^\infty(\mathbb{R})$
- $\frac{|\lambda|^s}{\deg(\mathbb{G})} > \sup_{z \in \mathbb{C}} \int \frac{\rho(v)}{|v-z|^s} dv =: C_s$, for some $0 < s < 1$

Then for any $E \in \mathbb{R}$, $\operatorname{Im}(z) > 0$, $x, y \in \mathbb{G}$
$$\mathbb{E}[|G(x,y;z)|^s] \le A \left(\frac{C_s \deg(\mathbb{G})}{|\lambda|^s}\right)^{d(x,y)} + \left(\frac{C_s}{|\lambda|^s} - A\right) \delta_{x,y}$$

$$A = \frac{C_s}{\deg(\mathbb{G}) \left(|\lambda|^s - C_s \deg(\mathbb{G})\right)}$$

proof. For each self-avoiding path $\hat\gamma$, using perturbation:
$$\langle \delta_{\hat\gamma(i)}, \frac{1}{H_i - z} \delta_{\hat\gamma(i)} \rangle = \frac{1}{\lambda V(\hat\gamma(i)) - \Gamma}$$

Γ: some quantity

$$\Rightarrow \quad \mathbb{E}[|(\cdot)|^s \mid V_{\neq \hat\gamma(i)}] \le \sup_{\Gamma \in \mathbb{C}} \int \frac{\rho(v)}{|\lambda v - \Gamma|^s} dv = \frac{C_s}{|\lambda|^s}$$

(Need $s<1$; otherwise diverges)

$$\Rightarrow \quad \mathbb{E}\left[\prod_{i=0}^{k} \left|\langle \delta_{\hat\gamma(i)}, \frac{1}{H_i - z} \delta_{\hat\gamma(i)} \rangle\right|^s\right] \le \left[\frac{C_s}{|\lambda|^s}\right]^{k+1}$$

$$\Rightarrow \quad \mathbb{E}[|G(x,y;z)|^s] \le \sum_{k=\operatorname{dist}(x,y)}^{\infty} \deg(\mathbb{G})^{k-1} \left[\frac{C_s}{|\lambda|^s}\right]^{k+1}$$

if $\operatorname{dist}(x,y) \ge 1$

if $\operatorname{dist}(x,y) = 0$, use