Localization criteria via correlator

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Beyond Simon-Wolff, another way to connect Green function & localization is through <u>eigenfunction correlator</u>

- H: self adj acting on $L^2(G)$.
- $I \subseteq \mathbb{R}$ Borel set

$$Q(x,y;I) = \sup_{\substack{F\in C(\mathbb{R})\\ \|F\|\le 1}} |\langle \delta_x, P_H(I) F(H)\delta_y\rangle| = \sup_{\substack{F\in C(\mathbb{R})\\ \|F\|\le 1}} \left|\int_I F(E)\, d\mu_{\delta_x,\delta_y}(E)\right| \le 1$$

$\downarrow$ complex values measure.

(For finite G, $Q(x,y;I) = \sum_{\substack{E\in I\\ \text{eigenvalue}}} |\varphi_E(x)\varphi_E(y)|$)
$\downarrow$ normalized eigenfunction.

Why useful?

- On one hand: $\sup_{t\in\mathbb{R}} |\langle\delta_x, P_H(I) e^{-itH}\delta_y\rangle| \le Q(x,y;I)$; decay of $Q(x,y;I)$ in $\operatorname{dist}(x,y)$ implies localization.
- On the other hand: $\mathbb{E}[Q(x,y;I)] \le C \liminf_{\eta\downarrow 0} \int_I \mathbb{E}[|G(x,y;E+i\eta)|^s]\, dE$.

We next elaborate on these points.

① <u>Def</u>. <u>Strong exponential dynamical localization in I</u>: for each $x\in G$

$$\sum_{\substack{y\in G\\ d(x,y)\ge R}} \mathbb{E}\left[\sup_t |\langle\delta_x, P_H(I) e^{-itH}\delta_y\rangle|^2\right] \le A e^{-\mu R}$$

for some $\mu>0$, $A<\infty$.

$\Rightarrow$ This implies, almost surely,

$$\sup_{t\in\mathbb{R}} \sum_{\substack{y\in G\\ d(x,y)\ge R}} |\langle\delta_x, P_H(I) e^{-itH}\delta_y\rangle|^2 \le A_x e^{-\mu_x R}$$

A_x, μ_x can be random, and depend on x.

(exponential dynamical localization)

- Dynamical localization implies spectral localization in I, by RAGE.
- More precisely, by RAGE, for each $\psi\in L^2(G)$,

$$\|P_H^c(I)\psi\|^2 = \lim_{L\to\infty}\lim_{T\to\infty}\frac{1}{T}\int_0^T \|1[G\setminus G_L] e^{-itH} P_H(I)\psi\|^2\, dt$$

Suffices to show RHS $=0$ for $\psi=\delta_x$.

Indeed, the integrand $= \sum_{y\notin G_L} |\langle\delta_y, e^{-itH} P_H(I)\delta_x\rangle|^2$

$$\le \sup_{t\in\mathbb{R}} \sum_{y\notin G_L} |\langle\delta_x, e^{-itH} P_H(I)\delta_y\rangle|^2$$

Then either: exponential DL for $\mu_x>0$, $A_x<\infty$

or strong exponential DL for $\mu=0$, $A<\infty$

or $\sum_{y\in G} Q(x,y;I)^2<\infty$, for all $x\in G$.

Suffices.

- Moreover, for each eigenvalue E, $|\langle\delta_x, P_H(\{E\})\delta_y\rangle| \le Q(x,y;E)$

$\Rightarrow$ if Q has exponential decay (in $d(x,y)$), so is every eigen function in the cyclic space of $\mathcal{H}_{H,\delta_y}$, with eigenvalue in I.

② Now we bound Q by factional moment of G.

Setup: $H=H_0+V$ act on $L^2(G)$, where the random potential $V: G\to\mathbb{R}$ satisfies that for each $x\in G$, the density of $V(x)\,|\,\{V(y)\}_{y\neq x}$, denoted by $\rho_x(\cdot\,|\,V_{\neq x})$, almost surely has $\sup_{v\in\mathbb{R}} (1+|v|^s)\rho_x(v\,|\,V_{\neq x}) \le C$, for some $s\in(0,1)$, $C<\infty$

<u>Then</u>: for any bounded open $I\subseteq\mathbb{R}$, $\mathbb{E}[Q(x,y;I)] \le C_s \liminf_{|\eta|\downarrow 0} \int_I \mathbb{E}|G(x,y;E+i\eta)|^s\, dE$

- This combined with ①, and fractional moment bound (on finite G), we get localization. (Strong Exponential DL)

$\Rightarrow$ Need one more step: $\mathbb{E}|G(x,y;z)|^s \le \liminf_{L\to\infty} \mathbb{E}|G_L(x,y;z)|^s$ $(*)$

Here $G_L\uparrow G$ are finite regular graphs, $G_L(x,y;z) = \langle\delta_x, (H_L-z)^{-1}\delta_y\rangle$. (think of H_L as acting on $L^2(\mathbb{Z}^d)$)

<u>proof of $(*)$</u>. For each $f=(H-z)\phi$, ϕ is finitely supported, we have $(H_L-z)^{-1}f = (H-z)^{-1}f$ for f large enough.

Note that $\{(H-z)\phi : \phi \text{ finite supp}\}$ is dense in $L^2(G)$, $\|(H_L-z)^{-1}\delta_y - (H-z)^{-1}\delta_y\|\to 0$ as $L\to\infty$; thus $G_L(x,y;z)\to G(x,y;z)$ as $L\to\infty$.

Then $(*)$ by Fatou's lemma.

Now we prove:

For $Q(x,y;I) = \sup_{\substack{F\in C_c(I)\\ \|F\|_\infty\le 1}} \left|\int F(E)\, d\mu_{\delta_x,\delta_y}(E)\right|$, since $F(E) = \lim_{\eta\downarrow 0}\frac{1}{\pi}\int F(E')\operatorname{Im}(E'-E-i\eta)^{-1}\, dE'$, we get

$$Q(x,y;I) = \sup_{\substack{F\in C_c(I)\\ \|F\|_\infty\le 1}} \left|\int \lim_{\eta\downarrow 0}\frac{1}{\pi}\int F(E')\operatorname{Im}(E'-E-i\eta)^{-1}\, dE'\, d\mu_{\delta_x\delta_y}(E)\right| \le \liminf_{\eta\downarrow 0}\frac{1}{\pi}\int_I \underbrace{\left|\int \operatorname{Im}(E'-E-i\eta)^{-1}\, d\mu_{\delta_x,\delta_y}(E)\right|}_{=|\langle\delta_x,\, \operatorname{Im}(-H+E'-i\eta)^{-1}\delta_y\rangle|} dE'$$

$$\Rightarrow \mathbb{E}\, Q(x,y;I) \le \liminf_{\eta\downarrow 0}\frac{1}{\pi}\int_I \mathbb{E}[|\langle\delta_x, \operatorname{Im}(H-E-i\eta)^{-1}\delta_y\rangle|]\, dE$$

$$\underset{\text{Cauchy-Schwarz}}{\le} \Big(\int_I \mathbb{E}\big[\underbrace{|\langle\delta_x, \operatorname{Im}(H-E-i\eta)^{-1}\delta_x\rangle|^{1-s}\cdot|\langle\delta_x, \operatorname{Im}(H-E-i\eta)^{-1}\delta_y\rangle|^s}_{(**)}\big]\, dE\Big)^{\frac{1}{2}}$$

$$\times\Big(\int_I \mathbb{E}\big[\underbrace{|\langle\delta_y, \operatorname{Im}(H-E-i\eta)^{-1}\delta_y\rangle|^{1-s}\cdot|\langle\delta_x, \operatorname{Im}(H-E-i\eta)^{-1}\delta_y\rangle|^s}_{(***)}\big]\, dE\Big)^{\frac{1}{2}}$$

(use that $|\langle\delta_x, \operatorname{Im}(H-E-i\eta)^{-1}\delta_y\rangle|^2 \le \langle\delta_x, \operatorname{Im}(H-E-i\eta)\delta_x\rangle\cdot\langle\delta_y, \operatorname{Im}(H-E-i\eta)^{-1}\delta_y\rangle$)

Note: $\langle\delta_x, \operatorname{Im}(H-z)^{-1}\delta_y\rangle = \frac{1}{2i}(G(x,y;z) - G(x,y;\bar z))$

$G_v(x,y;z) = \dfrac{G_0(x,y;z)}{1+v\,G_0(x,x;z)}$, Here G_v is the green function for $H+(v-V(x))|\delta_x\rangle\langle\delta_x|$.

Thus $(**)$ is bounded by:

$$\left(\frac{|\operatorname{Im}(G_0(x,x;z)^{-1})|}{|V(x)+G_0(x,x;z)^{-1}|^2}\right)^{1-s}\cdot\frac{|1+v\,G_0(x,x;z)|^s}{|1+V(x)\,G_0(x,x;z)|^s}|G_v(x,y;z)|^s$$

Here $v\in\mathbb{R}$ is arbitrary, $z=E+i\eta$

By taking expectation over $V(x)\,|\,\{V(y)\}_{y\neq x}$, and use that $\sup_{v\in\mathbb{R}}(1+|v|^s)\rho_x(v\,|\,V_{\neq x})\le C$, we conclude that (by Auxiliary lemma below)

$\mathbb{E}(**) \le C_s(|v|^s+1)\cdot|G_v(x,y;E+i\eta)|^s$

Note that this is for any v; by averaging over v with density $\sim (|v|^s+1)^{-1}\rho_x(v\,|\,V_{\neq x})$,

we get $\mathbb{E}(**) \le C_s\,\mathbb{E}|G_v(x,y;E+i\eta)|^s$.

Same for $(***)$.

<u>Auxiliary lemma</u>. for $\rho\ge 0$ with $\sup_{v\in\mathbb{R}}(1+|v|^s)\rho(v)<\infty$, for some $s\in(0,1)$, and $z\in\mathbb{C}\setminus\mathbb{R}$

$$(1+|z|^s)\int \frac{|\operatorname{Im} z|^{1-s}}{|v-z|^{2-2s}}\frac{\rho(v)}{|v-z|^s}\, dv \le C_s(\rho).$$

- Might be enlightening to consider finite G:

In this case, for any bounded $I\subseteq\mathbb{R}$,

$$Q(x,y;I) = \lim_{s\uparrow 1}\frac{1-s}{2}\int_I |G(x,y;E)|^s\, dE$$

<u>proof</u> It suffices to assume that I contains one eigenvalue E_*; and let $E_1,\dots,E_n$ be all eigenvalues.

Then $\text{RHS} = \lim_{s\uparrow 1}\frac{1-s}{2}\int_I\left|\sum_{m=1}^n (E-E_m)^{-1}\mu_{\delta_x\delta_y}(\{E_m\})\right|^s dE = |\mu_{\delta_x\delta_y}(\{E_*\})| = \text{LHS}.$