MSA near the edge: initial scale

Sunday, May 3, 2026 5:14 PM

Other than fraction moments, another approach to bound Green functions is the Multi-Scale Analysis (MSA) method.

- General idea: again bound Green function for $H = -\Delta + V$, restricted to some large boxes; do induction on the size of the boxes.
- Need induction base: fix box side, consider near edge/large disorder, bound the "density of eigenvalues" (Using e.g. Combes-Thomas, eigenvalue bounds $\Rightarrow$ Green function bounds)

We next consider eigenvalues near the edge.

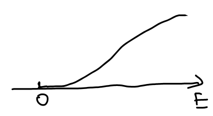
Lifshitz tail

Recall the DOS (density of states) measure: $\mu(I) = \mathbb{E}\langle \delta_x, P_H(I)\delta_x\rangle$ of (ergodic) operator H.

- For $H = -\Delta$, via Fourier transform, same as multiplying $2\sum_{i=1}^{d}(1-\cos\xi_i)$, on $L^2([0,2\pi]^d)$; $\mathcal{F}\delta_0 = 1$ constant function.

  For $E > 0$ small, $\mu([0,E]) = \left|\{\xi\in[0,2\pi]^d : 2\sum_{i=1}^{d}(1-\cos\xi_i) \le E\}\right| \cdot (2\pi)^{-d} \sim C\cdot E^{d/2}$
- For $H = -\Delta + V$, V iid random; $\mathrm{supp}(V(0)) \subset [0,\infty)$, Lifshitz speculated that $\mu([0,E]) \sim C_1 \exp(-C_2 E^{-\frac{d}{2}})$.

  Why? eigenvalue $E \Rightarrow$ eigenfunction like $\sin(E^{\frac{1}{2}}(\xi_1+\cdots+\xi_d))$; $V<E$ in a $(E^{-\frac{1}{2}})^d$ box.



- Can approximate DOS from finite dim:

  Split $\mathbb{Z}^d$ into L^d boxes; let
  $$\Delta_L^N \psi(x) = \sum_{\substack{y\sim x\\ x,y \text{ same box}}} (\psi(y)-\psi(x)) \qquad \text{Neumann boundary}$$
  $$\Delta_L^D \psi(x) = \sum_{\substack{y\sim x\\ x,y \text{ same box}}} (\psi(y)-\psi(x)) + \sum_{\substack{y\sim x\\ x,y \text{ diff box}}} (-2\psi(x)) \qquad \text{Dirichlet boundary}$$

  $\Rightarrow$ $\Delta - \Delta_L^D$, $\Delta_L^N - \Delta$ are <u>positive</u> operators

$\Rightarrow$ Can prove: For $H_L^D = -\Delta_L^D + V$, $H_L^N = -\Delta_L^N + V$,

$$\underbrace{\frac{1}{L^d}\mathbb{E}N(H_L^D,[0,E])}_{\#\text{ of eigenvalues in }[0,E]},\ \frac{1}{L^d}\mathbb{E}N(H_L^N,[0,E]) \to \mu([0,E])$$

(see: note on ergodic operators)

Moreover: $\frac{1}{L^d}\mathbb{E}N(H_L^D,[0,E]) \le \mu([0,E]) \le \frac{1}{L^d}\mathbb{E}N(H_L^N,[0,E])$

Below we think of H_L^D, H_L^N acting on $L^2([-L/2,L/2]^d)$; then

$$\frac{1}{L^d}\mathbb{P}[E_0(H_L^D)<E] \le \mu([0,E]) \le \mathbb{P}[E_0(H_L^N)<E]$$

- We next upper bound $\mathbb{P}[E_0(H_L^N)<E]$; for this, we need a lower bound of $E_0(H_L^N)$.

> Temple's inequality: for M being an Hermitian matrix, with the smallest two eigenvalues being $E_0<E_1$; and ψ satisfying $\|\psi\|=1$, $\langle\psi, M\psi\rangle < E_1$, we must have $E_0 \ge \langle\psi, M\psi\rangle - \dfrac{\langle M\psi, M\psi\rangle - \langle\psi, M\psi\rangle^2}{E_1 - \langle\psi, M\psi\rangle}$
>
> <u>Proof</u>. This simply follows from that $\langle\psi, (M-E_0)(M-E_1)\psi\rangle \ge 0$ for each ψ.

Take $\psi \equiv L^{-d/2}$ in the box. Then $\Delta_L^N\psi = 0$; and $\langle\psi, H_L^N\psi\rangle = L^{-\frac{d}{2}}\sum_x V(x)$, $\langle\psi, (H_L^N)^2\psi\rangle = L^{-\frac{d}{2}}\sum_x V(x)^2$

Need $\langle\psi, H_L^N\psi\rangle < E_1$: note that $E_1(H_L^N) \ge E_1(-\Delta_L^N) > c\cdot L^{-2}$; then consider $\tilde{H}_L^N := -\Delta_L^N + \tilde{V}$, where $\tilde{V} = \min(V, \frac{c}{3L^2})$; then $\langle\psi, \tilde{H}_L^N\psi\rangle \le \frac{c}{3L^2}$

$$\Rightarrow E_0(H_L^N) \ge E_0(\tilde{H}_L^N) \ge L^{-\frac{d}{2}}\sum_x \tilde{V}(x) - \frac{L^{-\frac{d}{2}}\sum_x \tilde{V}(x)^2}{\frac{2c}{3}L^{-2}} \ge \frac{L^{-\frac{d}{2}}}{2}\sum_x \tilde{V}(x)$$

$$\Rightarrow \mathbb{P}[E_0(H_L^N)<E] \le \mathbb{P}\left[\frac{L^{-\frac{d}{2}}}{2}\sum_x \tilde{V}(x) < E\right]$$

Take L s.t. $\frac{c}{6L^2} = E \Rightarrow$ above $\le \exp(-c'L^d) = \exp(-c'E^{-d/2})$

$\Rightarrow$ In conclusion: $\mu([0,E]) \le \mathbb{P}[E_0(H_L^N)<E] \le \exp(-c'E^{-d/2})$

> For MSA: we can also bound $\mathbb{P}[E_0(H_L^N)<L^{-\theta}]$ for $\theta\in(0,2)$:
>
> take $\ell = L^{\frac{\theta}{2}}$; split box $[-\frac{L}{2},\frac{L}{2}]^d$ into $(\frac{L}{\ell})^d$ many boxes with side length ℓ;
>
> $\Rightarrow E_0(H_L^N) \ge \min_k E_0(H_{\ell,k}^N)$
>
> $\Rightarrow \mathbb{P}[E_0(H_L^N)<L^{-\theta}] < L^{(1-\theta/2)d}\cdot\exp(-cL^{d\theta/2})$.

- For the complementary lower bound (for $\mu([0,E])$), need a bit assumption on the iid distribution of V.

  Namely, we want upper bound $E_0(H_L^D)$.

  We have $E_0(H_L^D) = \inf\limits_{\psi\in L^2([-L/2,L/2]^d)} \dfrac{\langle\psi, H_L^D\psi\rangle}{\|\psi\|_2^2} \le \inf\limits_{\psi} \dfrac{\langle\psi, -\Delta_L^D\psi\rangle}{\|\psi\|_2^2} + \max\limits_x V(x)$

  $$\langle\psi, -\Delta_L^D\psi\rangle \sim \sum_{x\sim y}|\psi(x)-\psi(y)|^2 + \sum_{x\in\partial\text{box}}|\psi(x)|^2$$

  To minimize the first term, take $\psi(x) = \frac{L}{2} - \|x\|_\infty$

  $\Rightarrow$ first term $= cL^{-2}$

  $$\mathbb{P}[\max_x V(x) < cL^{-2}] = \underbrace{\mathbb{P}[V(0)<cL^{-2}]}_{\underline{\text{assume}}\ > L^{-\xi} \text{ for some } \xi<\infty}{}^{L^d} = \exp(-c\log L\cdot L^d)$$

  $\Rightarrow \mathbb{P}[E_0(H_L^D) < cL^{-2}] \ge \exp(-c\log L\cdot L^d)$

  $\Rightarrow \mu([0,E]) \ge \exp(-c|\log(E)|\cdot E^{-d/2})$.

Combes-Thomas bound.

Let H be a self-adj operator acting on $L^2(\mathbb{G})$, with $\mathbb{G}$ countable. Then for z with $\Delta = \mathrm{dist}(z, \mathrm{Spec}(H)) > 0$, and $\mu>0$,

$$|G(x,y;z)| \le \frac{1}{\Delta - S_\mu}\exp(-\mu\, d(x,y)), \quad \text{if } \Delta > S_\mu.$$

Here $S_\mu := \sup_x \sum_y |H(x,y)|\,(e^{\mu d(x,y)}-1)$. (For $H = -\Delta+V$, $S_\mu = 2d(e^\mu - 1)$)

<u>Proof</u>. $G(x,y;z)e^{\mu d(x,y)} = \langle\delta_x, M_y(H-z)^{-1}M_y^{-1}\delta_y\rangle$, where $M_y(x) = \exp(\mu(d(x,y)\wedge R))$, R large

$= \langle\delta_x, (M_yHM_y^{-1} - z)^{-1}\delta_y\rangle$

For $B := M_yHM_y^{-1} - H$, $|B(x,w)| \le |H(x,w)|\cdot(\exp(\mu d(x,w))-1)$

$\Rightarrow \sum_w |B(x,w)| \le S_\mu$, $\sum_x |B(w,x)| \le S_\mu$, $\Rightarrow \|B\| \le S_\mu$.

$$\sum_x\Big(\sum_w B(x,w)\psi(w)\Big)^2 = \sum_x\sum_{w_1,w_2} B(x,w_1)B(x,w_2)\psi(w_1)\psi(w_2) \le \sum_x\sum_{w_1,w_2} B(x,w_1)B(x,w_2)\psi(w_1)^2 \le S_\mu^2\|\psi\|_2^2.$$

For $\Delta > \|B\|$, $\mathrm{dist}(z, \mathrm{Spec}(H+B)) > \Delta - \|B\|$, thus $\|(H+B-z)^{-1}\| \le \frac{1}{\Delta - \|B\|} \le \frac{1}{\Delta - S_\mu}$, and conclusion follows. $\square$

- For $H = -\Delta+V$, by taking $\mu = c\Delta$ we get $|G(x,y;z)| \le \frac{2}{\Delta}\exp(-c\Delta d(x,y))$
- Can improve: $B(x,w) = H(x,w)\cdot(\exp(\mu(\underbrace{d(y,w)\wedge R - d(y,x)\wedge R}_{s(x,w)}))-1)$

  write $B = C + iD$, where

  $C(x,w) = \frac{1}{2}(B(x,w) + B(w,x))$

  $D(x,w) = -\frac{i}{2}(B(x,w) - B(w,x))$

  C, D self-adj

  $|C(x,w)| \le c\mu^2|H(x,w)| \Rightarrow \|C\| \le c\mu^2$.

  $$\|(H+C+iD-z)^{-1}\| \le \frac{1}{\mathrm{dist}(\mathrm{Spec}(H+C), \mathrm{Re}\, z)} \le \frac{1}{\Delta_{Re} - c\mu^2}, \text{ if } \Delta_{Re} > c\mu^2$$

  $\Delta_{Re} = \mathrm{dist}(\mathrm{Re}\, z, \mathrm{Spec}\, H)$

  By taking $\mu = c\sqrt{\Delta_{Re}}$, we get

  $$|G(x,y;z)| \le \frac{2}{\Delta_{Re}}\exp(-c\sqrt{\Delta_{Re}}\, d(x,y))$$

> Write $X = H + C - \mathrm{Re}\, z$, $Y = D - \mathrm{Im}\, z$
>
> X positive
>
> $\Rightarrow (X+iY)^{-1} = (\sqrt{X})^{-1}(1 + i\underbrace{(\sqrt{X})^{-1}Y(\sqrt{X})^{-1}}_{\text{self-adj}})^{-1}(\sqrt{X})^{-1}$
>
> $\Rightarrow \|(X+iY)^{-1}\| \le \|(\sqrt{X})^{-1}\|\cdot 1\cdot\|(\sqrt{X})^{-1}\| = \|(\sqrt{X})^{-1}\|^2 = \mathrm{dist}(0, \mathrm{Spec}(X))^{-1}$