MSA induction step

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Geometric resolvent equation:

For $\Lambda_1 \subseteq \Lambda_2$, $x \in \Lambda_1$, $y \in \Lambda_2 \setminus \Lambda_1$, and $z \notin \operatorname{Spec} H_{\Lambda_1} \cup \operatorname{Spec} H_{\Lambda_2}$,

$$(H_{\Lambda_2}-z)^{-1}(x,y) = \sum_{\substack{(w,w')\in\partial\Lambda_1 \\ w\in\Lambda_1,\ w'\in\Lambda_2\setminus\Lambda_1}} (H_{\Lambda_1}-z)^{-1}(x,w)\ (H_{\Lambda_2}-z)^{-1}(w',y).$$

proof Write $H_{\Lambda_2} = H_{\Lambda_1} \oplus H_{\Lambda_2\setminus\Lambda_1} + \Gamma$, $\quad \Gamma(w,w') = \begin{cases} -1 & (w,w')\in\partial\Lambda_1 \\ 0 & \text{else} \end{cases}$

(slightly different Γ for Neumann/Dirichlet versions)

Then

$$(H_{\Lambda_2}-z)^{-1} = (H_{\Lambda_1}\oplus H_{\Lambda_2\setminus\Lambda_1}-z)^{-1} - (H_{\Lambda_1}\oplus H_{\Lambda_2\setminus\Lambda_1}-z)^{-1}\,\Gamma\,(H_{\Lambda_2}-z)^{-1}$$

at (x,y): $\quad \underset{\displaystyle 0}{\Downarrow} \qquad \underset{\displaystyle \underset{\underset{\displaystyle (x,w)}{\downarrow}}{(H_{\Lambda_1}-z)^{-1}} \oplus (H_{\Lambda_2\setminus\Lambda_1}-z)^{-1}}{\Downarrow} \qquad \underset{\displaystyle (w,w')}{\searrow} \qquad \underset{\displaystyle (w',y)}{\searrow}$

$\square$

(For Neumann/Dirichlet, slightly different form)

(The proof is for $z \notin \operatorname{Spec} H_{\Lambda_1} \cup \operatorname{Spec} H_{\Lambda_2} \cup \operatorname{Spec} H_{\Lambda_2\setminus\Lambda_1}$, but can do analytic continuation)

Now one can apply the above repeatedly: for a box of size L, and x near center, y at boundry,

$$(H_L-z)^{-1}(x,y) \le 2d\ell^{d-1}\underset{\displaystyle\text{for some } x_1 \text{ at the boundry of length } \ell \text{ box around } x}{(H_1-z)^{-1}(x,x_1)}(H_L-z)^{-1}(x_1,y) \quad \le (2d\ell^{d-1})^2\cdot(H_1-z)^{-1}(x,x_1)\,(H_2-z)^{-1}(x_1,x_2)\,(H_L-z)^{-1}(x_2,y) \le \cdots$$

H_1: operator H_L restricted to the size ℓ box

- If each $(H_1-z)^{-1}$, $(H_2-z)^{-1}$, … has exponential in distance decay, we get that $(H_L-z)^{-1}(x,y)$ is exponentially small.
- Here the last term $|(H_L-z)^{-1}(x_k,y)| \le \dfrac{1}{\operatorname{dist}(z, \operatorname{Spec}(H_L))}$; can be bounded by e.g. Wegner estimate (i.e., spectral averaging),
- Question: what if some small box Green function is not decaying?

  Idea: can still carry out if "most" small boxes are good.

> More precisely, by Wegner,
> $\mathbb{P}[\exists \text{ eigenvalue dist} < \theta \text{ from } z] \le C\cdot\theta\cdot L^d$
> $\Rightarrow$ With prob $> 1 - C\theta L^d$,
> $|(H_L-z)^{-1}(x,y)| \le \theta^{-1}$

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Def. The cube $\Lambda_L(x_0)$ of side length L is (γ,E)-good, if $|(H-E)^{-1}(x,y)| = |G(x,y;E)| \le e^{-\gamma L}$ for any $x\in\Lambda_{L^{\frac12}}(x_0)$ and $y \in \partial\Lambda_L(x_0)$.

Goal: For $L=\ell^\alpha$, $1<\alpha<2$, we want to show: for Λ_L, if most of its contained ℓ-sise boxes are (γ,E)-good, and H_L has no eigenvalue of $\text{dist}<e^{-\sqrt{L}}$ from E, we must have that Λ_L is (γ',E)-good.

- In the estimate above, where we assume each box of size ℓ is good, we can upper bound $(H_L-z)^{-1}(x,y)$ by

  $$\left((2d\ell^{d-1})e^{-\gamma\ell}\right)^k e^{\sqrt{L}}, \qquad k \ge \frac{L}{\ell}$$

  $$\Rightarrow \le e^{-\gamma L + \frac{L}{\ell}(d-1)\cdot\log\ell + \frac{L}{\ell}\log 2d + \sqrt{L}}$$

Take a slightly smaller $\gamma'<\gamma$ (ℓ is large enough, depending on γ. (e.g., $\gamma' < \gamma - \frac{(d-1)\log\ell}{\ell} - \frac{\log 2d}{\ell} - \frac{1}{\ell^{d/2}}$; does NOT $\to 0$ by repeately applying)

- Now, let's instead assume "at most one" bad γ-box:

  more precisely, assume no two disjoint bad γ-boxes,

  equivalently, each ℓ box with center outside a 2γ-box is good.

  Q: How to handle the 2ℓ-box?

  Again use geometric resolvent: $G_L(x,u;E) \le \sum_{\substack{w,w'\in\partial\Lambda_{2\ell} \\ w\sim w'}} \underset{\displaystyle\underset{\displaystyle\text{bound by } e^{\sqrt{2\ell}}, \text{ with probability } > 1-Ce^{-\sqrt{2\ell}}\cdot\ell^d, \text{ from Wegner estimate.}}{\downarrow}}{\underline{G_{2\ell}(u,w;E)}}\, G_L(w',x;E)$, for u in the bad 2ℓ-box.

  Next step: since w' outside the bad box, next step is exponentially small.

  $\Rightarrow$ Can still bound G_L, assuming "at most one bad ℓ box", and "bad 2ℓ-box is not too bad."
- Probabilistic estimate: If $\mathbb{P}[\ell\text{-box is }(\gamma,E)\text{-good}] \ge 1-\ell^{-p}$ for some big p:

  $\mathbb{P}[\text{"more than one } \ell \text{ bad" in } L\text{-box}] < L^{2d}\ell^{-2p}$

  Wegner: $\mathbb{P}[L\text{-box is "too bad"}] < e^{-\sqrt{L}}$

  $\mathbb{P}[\exists\, 2\ell\text{-box "too bad"}] < L^{2d}e^{-\sqrt{2\ell}}$

  $\Rightarrow$ sum $< L^{-p}$, if $\alpha p < -2d\alpha + 2p$, ℓ large enough.

Now, start from some ℓ_0, $\mathbb{P}[L=\ell_0^{\alpha^k}\text{-box is }(\gamma_k,E)\text{-good}] \ge 1-L^{-p}$, for all $k\in\mathbb{N}$. $(*)$

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From MSA result $(*)$, how to get localization?

Consider generalized eigenfunctions ψ with generalized eigenvalue E

i.e. $H\psi = E\psi$, and $|\psi(x)| \le (|x|+1)^m$ for some $m>0$. (At most polynomial growth)

Note that $H = H_\Lambda \oplus H_{\Lambda^c} + \Gamma$, in Λ we have $(H_\Lambda - E)\psi|_\Lambda = -\Gamma(\psi|_\Lambda)$, or $\psi|_\Lambda = -(H_\Lambda-E)^{-1}\Gamma(\psi|_\Lambda)$

$\Rightarrow$ for $x\in\Lambda$, $\psi(x) = -\sum_{\substack{w,w'\in\partial\Lambda \\ w\sim w'}} G_\Lambda(w,x;E)\,\psi(w')$

Lemma. If $\Lambda_L = \Lambda_{\ell_0^{\alpha^k}}$ is (γ_∞,E)-good for some $\gamma_\infty>0$, and all $k\in\mathbb{N}$ large enough, we must have that E is not a generalized eigenvalue.

proof Suppose not true, for generalised eigenfunction ψ, we can use goodness to show that $\psi(x)=0$ for any $x\in\mathbb{Z}^d$, by taking large enough boxes.

Theorem The spectrum of H equals the closure of the set of generalised eigenvalues; moreover, $\operatorname{Spec}(H)\setminus\overline{\text{generalized eigenvalues}}$ has zero spectral measure.

Assuming it, we can prove no ac. spectrum, near the edge.

Namely, we have that for each $E\in[0,E_0]$, a.s. E is not a generalice eigenvalue.

$\Rightarrow$ by Fubini, a.s., set of generalized eigenvalues in $[0,E_0]$ has zero-Lebesgue measure.

$\Rightarrow$ a.s., $\mu^{ac}_{\delta_x}([0,E_0]) = \mu^{ac}_{\delta_x}([0,E_0]\cap\{\text{gen. e.v.}\}) + \mu^{ac}_{\delta_x}([0,E_0]\setminus\{\text{gen. e.v.}\}) = 0$.

- How to remove sc component?

  Need use spectrum averaging, to get that generalized eigenvalues are zero-spectral measure.

  More precisely we aim to show $\mathbb{E}\mu^c_{\delta_x}([0,E_0])=0$, for any fixed x.

  Define K to be the set of all $E\in[0,E_0]$, satisfying "structure decay bounds".

Suppose:
- $E\in K \Rightarrow E$ is not generalized eigenvalue, unless E is eigenvalue in L^2.
- $\mathbb{P}[E\in K]=1$ for each $E\in[0,E_0]$
- Resampling $V(x)$ does not affect K.

$$\Rightarrow \underset{\displaystyle\underset{\displaystyle \mathbb{E} \text{ w.r.t. } V(x)}{\downarrow}}{\mathbb{E}}\mu^c_{\delta_x}([0,E_0]) \underset{\displaystyle\underset{\displaystyle\text{because } \mu^c_{\delta_x}(K)=0, \text{ a.s., by } \underline{\text{Theorem}}}{\searrow}}{=} \mathbb{E}\mu^c_{\delta_x}([0,E_0]\setminus K) \underset{\displaystyle\underset{\displaystyle\text{by spectrum averaging}}{\downarrow}}{\le} C\cdot|[0,E_0]\setminus K| = 0 \quad \text{a.s.}$$

$\Rightarrow$ a.s., $\mu^c_{\delta_x}([0,E_0])=0$.

The "structure decay bounds" can be the following: for each large enough $k\in\mathbb{N}$, $L=\ell_0^{\alpha^k}$, $\Lambda_L(x)$ is (γ_k,E)-good, for any $x\in\Lambda_{3L}(0)\setminus\Lambda_{2L}(0)$.

(Then for any generalized eigenfunction ψ with $H\psi=E\psi$, must have exponential decay, therefore $\psi\in L^2(\mathbb{Z}^d)$.)

($\mathbb{P}[E\in K]=1$ by MSA)

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It now remains to prove Theorem on generalized eigenvalues.

① We first show that, for λ a.e. under spectral measure, λ is generalized eigenvalue.

Here: spectral measure means every spectral measure.

It is convenient to consider $\rho = \sum_x a(x)\,\mu_{\delta_x}$, where each $a(x)>0$, $\sum_x a(x)=1$.

$\Rightarrow \mu_{\delta_x,\delta_y} \ll \rho$ for any $x,y\in\mathbb{Z}^d$.

Fix any $y\in\mathbb{Z}^d$, consider $f_{y,\lambda}(x) := \dfrac{d\mu_{\delta_x,\delta_y}}{d\rho}(\lambda)$, for λ. a.e. under ρ.

Idea: $f_{y,\lambda}$ is a generalized eigenfunction (for λ a.e. $\sim\rho$)

Indeed, for any g smooth with compact support,

$$\int \lambda g(\lambda) f_{y,\lambda}(x)\,d\rho(\lambda) = \langle \delta_x, Hg(H)\delta_y\rangle = \langle H\delta_x, g(H)\delta_y\rangle = \langle V(x)\delta_x, g(H)\delta_y\rangle - \sum_{w\sim x}\langle\delta_w, g(H)\delta_y\rangle$$

$$= \int V(x)\,g(\lambda) f_{y,\lambda}(x)\,d\rho(\lambda) - \sum_{w\sim x}\int g(\lambda) f_{y,\lambda}(w)\,d\rho(\lambda) = \int g(\lambda)\,(Hf_{y,\lambda})(x)\,d\rho(\lambda)$$

$\Rightarrow$ for λ a.e. $\sim\rho$, $\lambda f_{y,\lambda}(x) = (Hf_{y,\lambda})(x)$.

(since $\mathbb{Z}^d$ countable, this holds simultaneously for all $x\in\mathbb{Z}^d$, for λ a.e. $\sim\rho$)

By Cauchy-Schwartz, $|f_{y,\lambda}(x)| \le \sqrt{\dfrac{d\mu_{\delta_x}}{d\rho}(\lambda)\cdot\dfrac{d\mu_{\delta_y}}{d\rho}(\lambda)} \le \dfrac{1}{\sqrt{a(x)a(y)}}$

(because $|\mu_{\delta_x,\delta_y}(I)| \le \sqrt{\mu_{\delta_x}(I)\,\mu_{\delta_y}(I)}$; see RAGE theorem notes)

By choosing $a(x)$ appropriately, can have $f_{y,\lambda}$ at most polynomial growth.

② We need also show that any generalised eigenvalue is in the spectrum.

Indeed, for $H\psi=\lambda\psi$, consider ψ_L, the restriction of ψ in a box of side length L.

$\|(H-\lambda)\psi_L\|^2 \le \|\psi_{L+1}\|^2 - \|\psi_{L-2}\|^2$; since ψ grows at most polynomically, $\dfrac{\|\psi_{L+1}\|^2}{\|\psi_{L-2}\|^2} \to 1$ as $L\to\infty$ along a subsequence.

(Otherwise, $\|\psi_{L+3k}\|^2 \ge (1+\theta)^k\|\psi_L\|^2$ for some $\theta>0$, for all k large enough)

$\Rightarrow \dfrac{\|(H-\lambda)\psi_L\|^2}{\|\psi_L\|^2} \to 0$ as $L\to\infty$ along some subsequence, $(H-\lambda)$ not invertible.

③ Finally, by ② we have $\overline{\{\text{gen. eigenvalue}\}} \subseteq \operatorname{Spec}(H)$

by ① we have $\rho(\{\text{gen. eigenvalues}\}^c)=0$, $\Rightarrow$ interior of the compliment is disjoint from $\operatorname{supp}(\rho)$

$\Rightarrow \operatorname{supp}(\rho) \subseteq \overline{\{\text{gen. eigenvalue}\}}$

$\Rightarrow$ since $\operatorname{Spec}(H) = \operatorname{supp}(\rho)$, we must have $\overline{\{\text{gen. eigenvalue}\}} = \operatorname{Spec}(H)$.