Airy$_\beta$ line ensemble

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β ensemble

(Dyson, 62') a distribution on $\{(x_1, \dots, x_N) \colon x_1 \le \cdots \le x_N\}$

$$\mathbb{P}[(\lambda_1, \dots, \lambda_N) = (x_1, \dots, x_N)] = \frac{1}{Z} \prod_{1 \le i < j \le N} |x_i - x_j|^\beta \prod_{i=1}^{N} W(x_i)$$

In mathematical physics Coulomb log-gas, β is the inverse temperature;

originally: energy levels of heavy nuclei

quantum physics and integrable systems: Calogero-Moser-Sutherland model, Selberg integral, orthogonal polynomials…

Higher dim Coulomb gas *XY model, Ginzburg-Landau, Laughlin wavefunction in fractional quantum Hall effect, etc.*

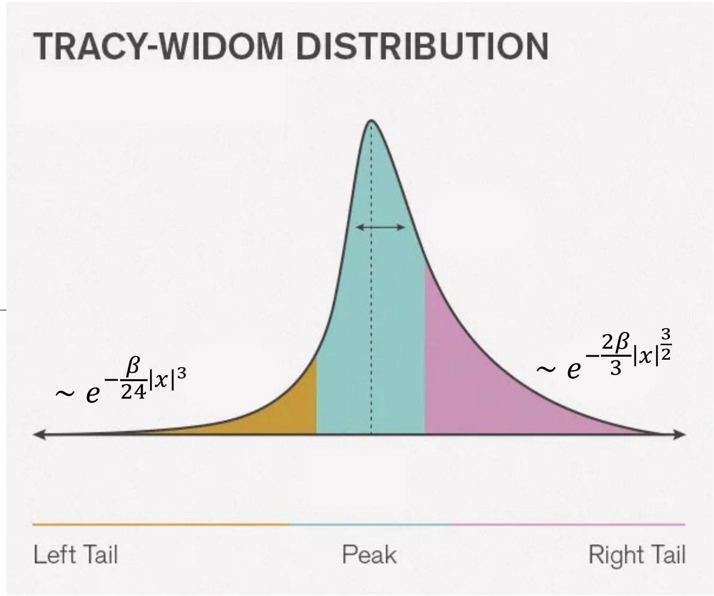
In number theory zeros of Riemann ζ function

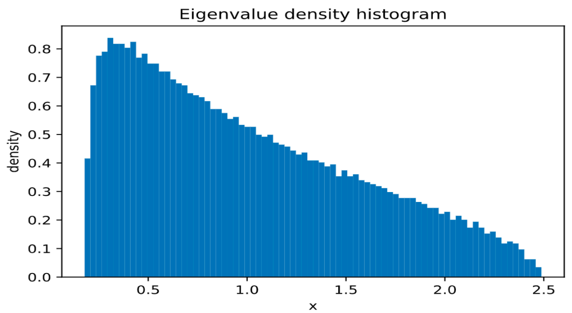
In probability/statistics eigenvalues of classical random matrices ($\beta = 1, 2, 4$)

Hermitian matrix ($X + X^*$)	$W(x) = e^{-x^2}$	(Gaussian β ensemble)
Wishart matrix (XX^*)	$W(x) = x^p e^{-x}$	(Laguerre β ensemble)
MANOVA matrix ($XX^*(XX^* + YY^*)^{-1}$)	$W(x) = x^p (1-x)^q$	(Jacobi β ensemble)

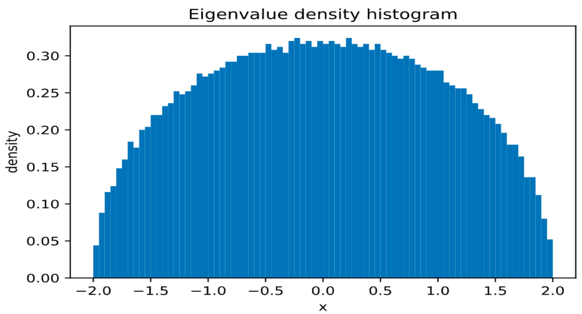
- $\beta = 1, 2, 4$: real, complex, quaternion entries
- General β: tri-diagonal matrix model (Dumitriu-Edelman, 02')

Limit of top particle: Tracy-Widom$_\beta$ distribution





Laguerre β ensemble: Marchenko–Pastur law



Gaussian β ensemble: semi-circle law

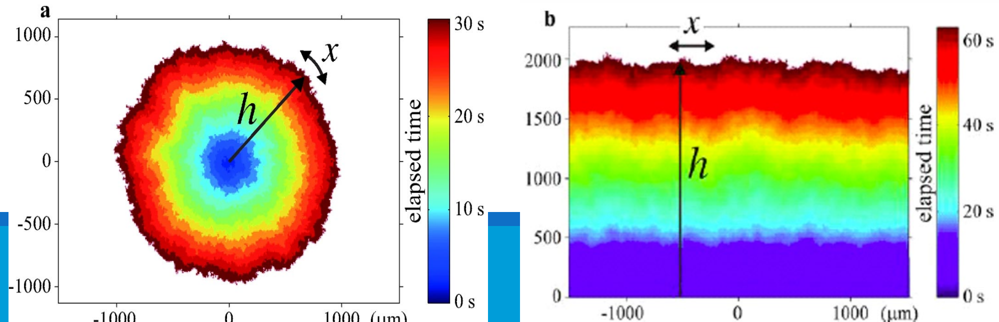
However, the largest eigenvalue has the same scaling limit: $\lim_{N\to\infty} \frac{(\lambda_1 - cN^a)}{N^b} = \textbf{Tracy-Widom}_{\boldsymbol{\beta}}$

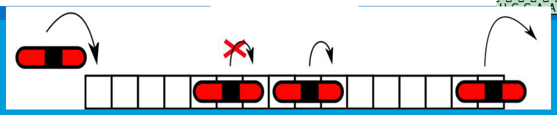
Universality Tracy-Widom$_\beta$ limit holds for most β ensembles (even some variants, e.g., discrete ones)

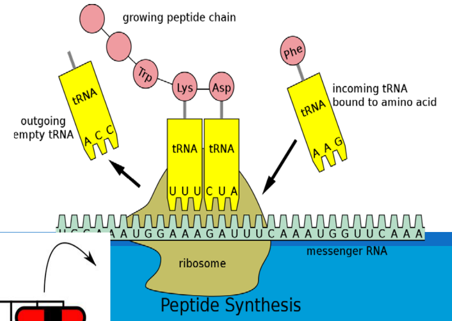
It also appears (or is conjectured) in many probabilistic models

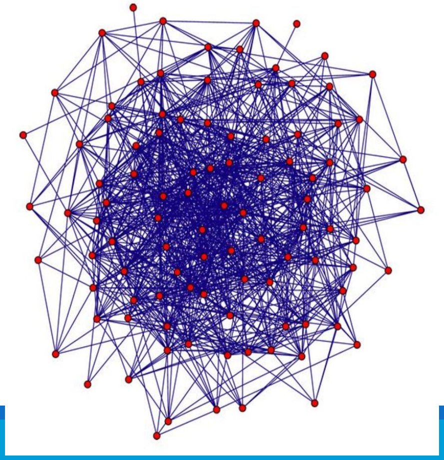
KPZ growth model and particle systems ($\beta = 1, 2$) *Adjacency matrix of random graph ($\beta = 1$)*

(general β) non-intersecting random walks, random partition, etc.

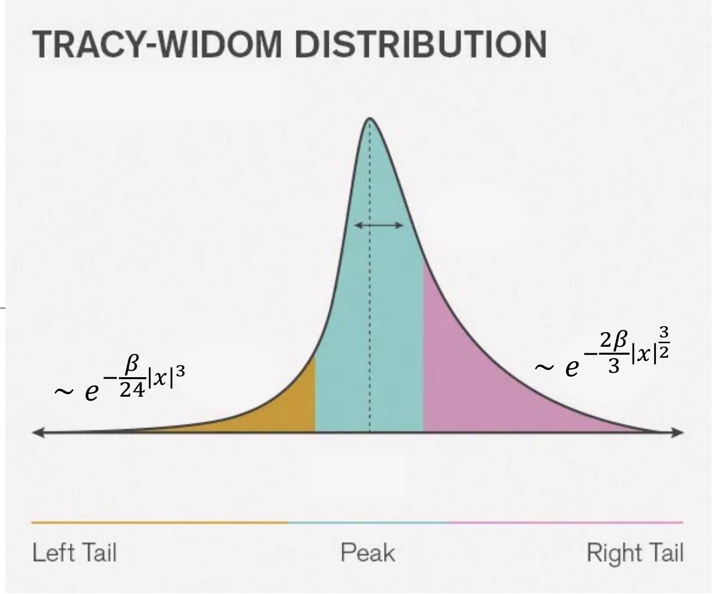








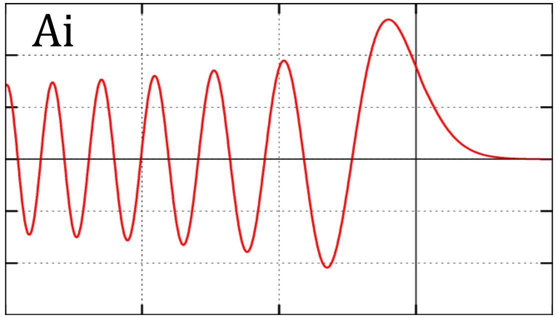
Tracy-Widom$_\beta$ distribution



History First for $\beta = 1,2,4$ (90s, by Tracy-Widom, etc.) using special structures
then general β (by Ramírez-Rider-Virag, 06') using tri-diagonal matrix

Airy$_\beta$ point process Not only λ_1, also for the first a few particles/eigenvalues ($\lambda_1 \geq \lambda_2 \geq \cdots$), which jointly converge to a point process (edge limit)

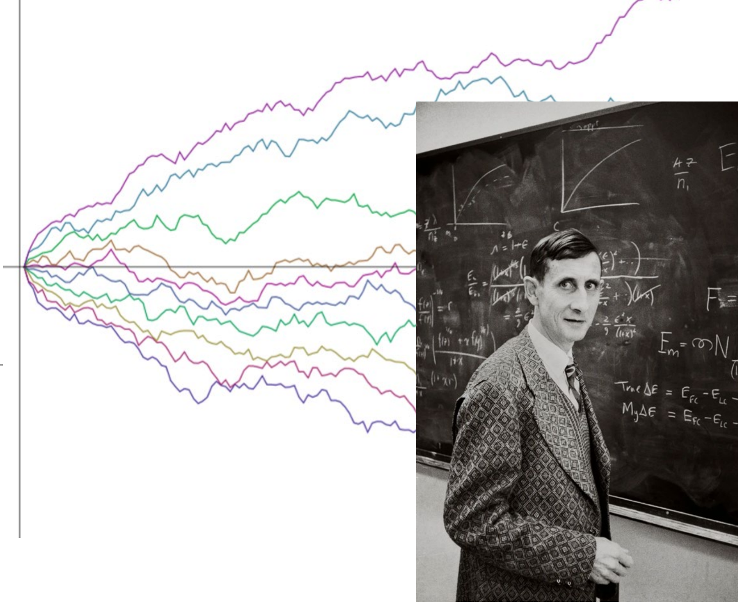
'Airy' in its name comes from the connection to the Airy function Ai, which solves the ODE $\mathrm{Ai}''(x) = x\mathrm{Ai}(x)$
e.g. when $\beta \to \infty$, the point process converges to zeros of Ai.



What we do: add a 'time' coordinate (Gaussian → Brownian motion)

- Many natural 'multi-time' extensions of β ensembles
- To better understand Tracy-Widom$_\beta$

Dyson Brownian motion (DBM)



(Dyson, 62') Dynamics: a diffusion in $\{(x_1, \dots, x_N) : x_1 \le \cdots \le x_N\}$

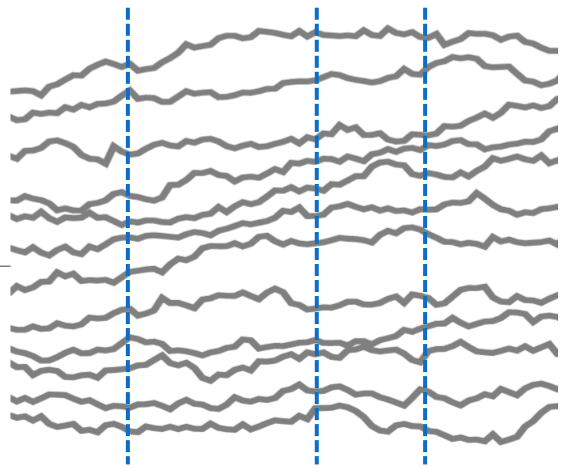
$$\mathrm{d}Y_i(t) = \frac{\beta}{2} \sum_{j \neq i} \frac{\mathrm{d}t}{Y_i(t) - Y_j(t)} + \mathrm{d}B_i(t), \qquad \forall 1 \le i \le N$$

Starting from zero, $(Y_1(t), \dots, Y_N(t))$ is Gaussian β ensemble $\prod_{1 \le i < j \le N} |x_i - x_j|^\beta \prod_{i=1}^N e^{-x_i^2/(2t)}$

The idea is to generalize the notion of matrix ensemble in such a way that the Coulomb gas model acquires a meaning, not only as a static model in timeless thermodynamical equilibrium, but as a dynamical system which may be in an arbitrary nonequilibrium state changing with time. ***Dyson, 62'***

- $\beta = 1,2,4$: eigenvalues of $A + (X_t + X_t^*)$, with $(X_t)_{ij}$ being independent Brownian motions,
  = DBM starting from $\mathrm{Spec}(A)$
  (used to prove universality of eigenvalue statistics, in e.g., Johansson, 00'; Erdos-Schlein-Yau, 09'; see also *A Dynamical Approach to Random Matrix Theory* by Erdos-Yau, 17')
- $\beta = 2$: central tool in KPZ universality class (*the directed landscape, Dauvergne-Virag-Ortman, 18'*)
- Driving function for multiple SLE$_\kappa$, with $\kappa = \frac{8}{\beta}$ *(since Cardy, 03'); e.g., Ising* $\kappa = 3$, $\beta = \frac{8}{3}$*; self-avoiding walk* $\kappa = \frac{8}{3}$, $\beta = 3$

Airy$_\beta$ line ensemble



(Gorin-Xu-Z. 24') For any $\beta > 0$, there is a unique ordered family of random processes, stationary and continuous in t, denoted by $\left\{\mathcal{A}_i^\beta(t)\right\}_{i=1}^\infty$, such that for any $\vec{\alpha} \in \mathbb{R}_+^m$ and $\vec{t} \in \mathbb{R}^m$,

$$\mathbb{E}\left[\prod_{j=1}^{m}\left(\sum_{i=1}^{\infty} \exp\left(\alpha_j \mathcal{A}_i^\beta(t_j)\right)\right)\right] = L_\beta(\vec{\alpha}, \vec{t}).$$

($L_\beta(\vec{\alpha}, \vec{t})$ to be defined later)

A new 'universal' object

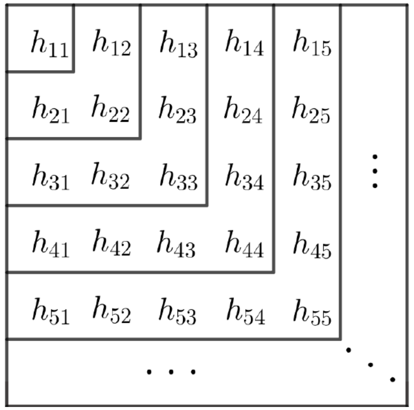
We call $\left\{\mathcal{A}_i^\beta(t)\right\}_{i=1}^\infty$ the Airy$_\beta$ line ensemble *(determined by Laplace transforms)*

(Gorin-Xu-Z., 24') Airy$_\beta$ line ensemble is the edge limit of (zero initial) DBM

i.e., $\lim_{N\to\infty} \frac{1}{2N^{1/3}}\left(Y_i\left(\frac{2N}{\beta} + \frac{2tN^{2/3}}{\beta}\right) - 2\sqrt{N(N+tN^{2/3})}\right)$

Other than DBM: Gaussian corners process

Hermitian matrix $X+X^*$ (eigenvalues: Gaussian $\beta=1,2,4$ ensemble), take corners

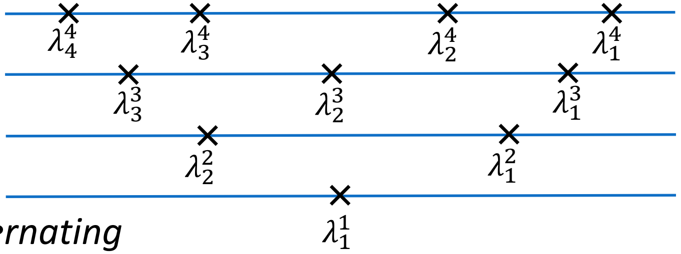


Joint law of eigenvalues: interlace

$$\mathbb{P}[\{\lambda_i^k\}_{1\le i\le k\le N}=\{x_i^k\}_{1\le i\le k\le N}]=\frac{1}{Z}\prod_{k=1}^{N-1}\prod_{1\le i<j\le k}|x_i^k-x_j^k|^{2-\beta}\prod_{a=1}^{k}\prod_{b=1}^{k+1}|x_a^k-x_b^{k+1}|^{\frac{\beta}{2}-1}\prod_{i=1}^{N}e^{-\frac{(x_i^N)^2}{2}}$$

Gaussian $\boldsymbol{\beta}$ corners process

(Okounkov-Olshanski, 97', Neretin, 03')



$\beta=2$: uniform interlacing particles; appear in statistical physics

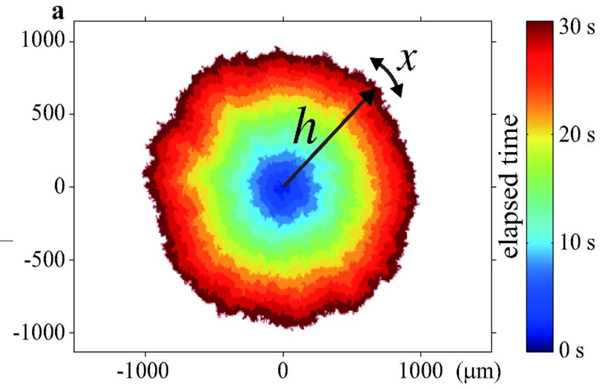
(dimers, Okounkov-Reshetikhin, 06', Johansson-Nordenstam, 06'; six-vertex/alternating sign matrices, Gorin-Greta, 13', Gorin 13')

(Gorin-Xu-Z., 24') Airy$_\beta$ line ensemble is the edge limit of Gaussian corners process

i.e., $\lim_{N\to\infty}\frac{N^{1/6}}{\sqrt{2\beta}}\left(\lambda_i^{N-tN^{2/3}}-\sqrt{2\beta(N-tN^{2/3})}\right)$

History of line ensembles and our approach

Top line $\mathcal{A}_1^{\beta=2}$ describes the boundary of KPZ growth



$\beta = 2$: Airy line ensemble, central in KPZ (through RSK correspondence)
(formulas in Prahofer-Spohn, 01'; continuity by Corwin-Hammond, 11')

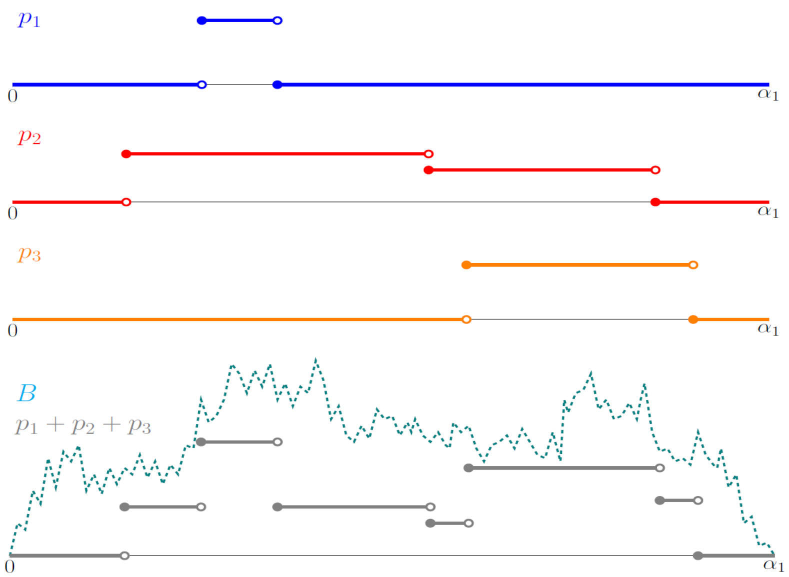
Much less known for other β: less structure; and tri-diagonal matrix does not extend

- (Sodin, 13') $\beta = 1,2,4$ using Hermitian matrix model; convergence for corners and DBM
- (Landon, 20') $\beta \geq 1$, convergence of DBM, Cauchy sequence arguments
- (Gorin–Kleptsyn, 21') $\beta = \infty$, distribution formulas, limit of corners and DBM ($\beta, N \to \infty$ simultaneously)

Our approach extract moments of DBM/corners using Dunkl differential operators acting on multivariate Bessel generating functions

i.e., $\mathcal{D}_{\mathrm{i}} = \frac{\partial}{\partial x_i} + \frac{\beta}{2}\sum_{j\neq i}\frac{1-\sigma_{ij}}{x_i - x_j}$ acting on $\mathbb{E}\left[\mathcal{B}_{Y_1(t),\dots,Y_N(t)}(x_1,\dots,x_N;\beta)\right] = \exp\left(\frac{t}{2}\sum_{i=1}^N x_i^2\right)$

The Laplace formula

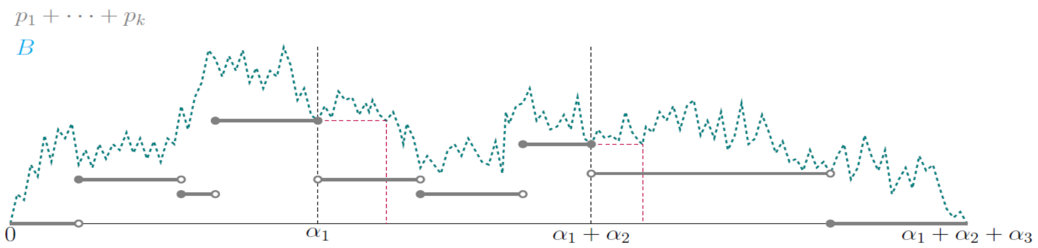


$$\mathbb{E}\left[\prod_{j=1}^{m}\left(\sum_{i=1}^{\infty}\exp\left(\alpha_j \mathcal{A}_i^{\beta}(t_j)\right)\right)\right] = L_\beta(\vec{\alpha}, \vec{t})$$

First moment ($m = 1$)

- Blocks: finitely many piece-wise constant functions $\{p_i\}_{i=1}^k$ on $[0, \alpha_1]$
- Brownian excursion $B\colon [0, \alpha_1] \to \mathbb{R}_{\geq 0}$

$$\int_\Omega \operatorname{sgn}(\{p_i\}_{i=1}^k)\, \mathbb{E}\left[\exp\left(\int_0^{\alpha_1} B(t) - \sum_{i=1}^{k} p_i(t)\,\mathrm{d}t\right)\mathbf{1}[B \in \text{constraints}]\right] \mathrm{d}\mu(\{p_i\}_{i=1}^k$$



Higher moments (assuming $t_1 \leq \cdots \leq t_m$, WLOG)

$\{p_i\}_{i=1}^k$ and B are on $[0, \alpha_1 + \cdots + \alpha_m]$; B is a concatenation of Brownian bridges, under some conditioning

$$\int_\Omega \operatorname{sgn}(\{p_i\}_{i=1}^k) \exp\left(\sum_{i=1}^{m-1}(t_i - t_{i+1})B(\alpha_1 + \cdots + \alpha_i)/2\right)\mathbb{E}\left[\exp\left(\int_0^{\alpha_1+\cdots+\alpha_m} B(t) - \sum_{i=1}^{k} p_i(t)\,\mathrm{d}t\right)\mathbf{1}[B \in \text{constraints}]\right] \mathrm{d}\mu(\{p_i\}_{i=1}^k)$$

More on universality

Beyond DBM and Gaussian β corners, many processes are expected to converge to the Airy$_\beta$ line ensemble

Some **continuous time** diffusions:

- DBM with general potential $\mathrm{d}Y_i(t) = \frac{\beta}{2}\sum_{j\neq i}\frac{\mathrm{d}t}{Y_i(t)-Y_j(t)} + V'(Y_i(t))\mathrm{d}t + \mathrm{d}B_i(t)$ (Langevin dynamics of β ensemble)
- Laguerre process (König-O'Connell, 01') $\mathrm{d}Y_i(t) = \left(n + \sum_{j\neq i}\frac{Y_i(t)+Y_j(t)}{Y_i(t)-Y_j(t)}\right)\mathrm{d}t + 2\sqrt{\frac{Y_i(t)}{\beta}}\mathrm{d}B_i(t)$

($\beta = 1,2,4$: eigenvalues of $X_tX_t^*$, with $(X_t)_{ij}$ being independent Brownian motions)

- Jacobi process (Demni, 09') $\mathrm{d}Y_i(t) = \left(p - mY_i(t) + \sum_{j\neq i}\frac{Y_i(t)-Y_i^2(t)+Y_j(t)-Y_j^2(t)}{Y_i(t)-Y_j(t)}\right)\mathrm{d}t + 2\sqrt{\frac{Y_i(t)(1-Y_i(t))}{\beta}}\mathrm{d}B_i(t)$

($\beta = 1,2,4$: Brownian motions in orthogonal/unitary group)

Some **discrete** models:

- Laguerre/Jacobi corners process: 'corners' in Wishart/MANOVA matrices

  for Jacobi, eigenvalues $(XX^*\left(XX^*+Y_{[k]}Y_{[k]}^*\right)^{-1})$, with $Y_{[k]}$= first k columns of Y (Borodin-Gorin, 13'; Sun, 16')

- More general interlacing sequences (Gelfand-Tsetlin Patterns): non-intersecting random walks, tiling, polymer, etc.
- Macdonald processes (Borodin-Corwin, 14'), and other general object in integrable probability

Towards universality: characterization

(Huang-Z. 24') Any $\{\lambda_i(t)\}_{i=1}^{\infty}$ must be the Airy$_\beta$ line ensemble, if the followings hold:

- ❑ $\lambda_1(t)$ is tight in t
- ❑ Take $S_t(z) = \sum_{i=1}^{\infty} \frac{1}{\lambda_i(t)-z} - \frac{1}{a_i-z}$. Then $\left|S_t(z) - \sqrt{z}\right| < C(t)\frac{\mathrm{Im}[\sqrt{z}]^{1-\delta}}{\mathrm{Im}[z]}$, for z away from $\mathbb{R}$
- ❑ $\mathrm{d}S_t(z) = \left(\frac{2-\beta}{2\beta}\partial_z^2 S_t(z) + \frac{1}{2}\partial_z S_t^2(z) - \frac{1}{2}\right)\mathrm{d}t + \mathrm{d}M_t(z)$, where $M_t(z)$ is the Martingale part, with quadratic variation $\mathrm{d}\langle M_t(z), M_t(w)\rangle = \frac{2}{\beta}\partial_z\partial_w \frac{S_t(z)-S_t(w)}{z-w}\mathrm{d}t$ *Dyson Brownian motion + Ito's formula*

Why/How to prove this?

General idea: think about Airy$_\beta$ line ensemble as infinite dimensional DBM, show 'uniqueness' of solution

For this, show **mixing**: two sets of particles under DBM get closer in time (appropriately coupled)

Many *issues*: infinite dimensional SDEs are not well-understood

Truncation? Then how to control boundaries?
May be shift of each other?

Solution Stieltjes transformation *(In its proof, eventually show poles get closer in time.)*

Towards universality

(Huang-Z. 24') Any $\{\lambda_i(t)\}_{i=1}^{\infty}$ must be the Airy$_\beta$ line ensemble, if the followings hold:

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To apply it for convergence: (1) check tightness (2) verify SDE

Continuous time diffusions:

(Huang-Z. 24') The edge limit of DBM with general potential, Laguerre process, and Jacobi process is the Airy$_\beta$ line ensemble.

Discrete models: SDE from Markovian/Gibbs structure

Tightness? *(may from dynamical loop equation, Gorin-Huang, 22')*