Shift-Invariance of the Colored TASEP

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The models: TASEP, colors, six-vertex
Totally Asymmetric Simple Exclusion Process (TASEP), and growing surface:
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Rotate by $\frac{\pi}{4}$, this corresponds to a corner growth process:
Totally Asymmetric Simple Exclusion Process (TASEP), and growing surface:

Rotate by $\frac{\pi}{4}$, this corresponds to a corner growth process:
TASEP and LPP

TASEP with step initial configuration also corresponds to Last Passage Percolation (LPP) with fixed starting point.

\[ \xi(v) \sim \text{Exp}(1), \text{i.i.d. } \forall v \in \mathbb{Z}^2 \]

Passage time:
\[ L_{u,v} := \max_{\gamma} \sum_{w \in \gamma} \xi(w) \]

LPP on \( \mathbb{Z}^2 \):

- \( \xi(v) \sim \text{Exp}(1), \text{i.i.d. } \forall v \in \mathbb{Z}^2 \)
- Passage time: \( L_{u,v} := \max_{\gamma} \sum_{w \in \gamma} \xi(w) \)
Known Results on LPP/Corner growth

- $L_{(0,0),(n,n)} \sim 4n$ (Rost, 1981).
- $2^{-4/3} n^{-1/3}(L_{(0,0),(n,n)} - 4n)$ converges weakly to the GUE Tracy-Widom distribution (Johansson, 2000).
- Point to line profile (Borodin and Ferrari, 2008)
  \[
  2^{-4/3} n^{-1/3} \left( L_{(0,0),(n-x(2n)^{2/3},n+x(2n)^{2/3})} - 4n \right) \Rightarrow \mathcal{A}_2(x) - x^2
  \]
  $\mathcal{A}_2$ is stationary and absolute continuous with respect to Brownian motion (Corwin and Hammond, 2014).
- KPZ fixed point (Matetski, Quastel, and Remenik, 2017)
  Airy sheet (Dauvergne, Ortmann, and Virág, 2018).
TASEP with colors

One particle at each integer, and the particle at $i$ is labeled $i$.
TASEP with colors

One particle at each integer, and the particle at \( i \) is labeled \( i \).

\[
\begin{array}{ccccccccc}
-9 & -8 & -7 & -6 & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9
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Rule of update: if \( a < b \), then with rate 1:

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Alternative description: a family of coupled step initial TASEPs, by considering all particles \( \leq i \).
TASEP with colors

One particle at each integer, and the particle at $i$ is labeled $i$.

Rule of update: if $a < b$, then with rate 1:

$$\begin{array}{c}
\text{a} & \text{b} \\
\rightarrow \\
\text{b} & \text{a}
\end{array}$$

but

$$\begin{array}{c}
\text{b} & \text{a} \\
\times \\
\rightarrow \\
\text{a} & \text{b}
\end{array}$$

Alternative description: a family of coupled step initial TASEPs, by considering all particles $\leq i$.
A general model in integrable probability (figures from Vadim):

- Color 7 (purple)
- Color 6 (blue)
- Color 5 (cyan)
- Color 4 (green)
- Color 3 (yellow)
- Color 2 (orange)
- Color 1 (red)
- Color 0 (gray)

Legend:

- \( \square \) = smaller color
- \( \square \) = larger color

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LPP geodesic environment
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= smaller color  
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Symmetries for the colored TASEP

Let $\zeta_t : \mathbb{Z} \to \mathbb{Z}$ be the configuration of the colored TASEP at time $t$. In particular, $\zeta_0$ is the identity map.

The following has the same distribution as $\zeta_t$:

- $x \mapsto \zeta_t(x - y) + y$ for any $y \in \mathbb{Z}$
- $x \mapsto -\zeta_t(-x)$
- $\zeta_t^{-1}$ (color-to-position symmetry, see e.g. Amir, Angel, and Valkó, 2011; Angel, Holroyd, and Romik, 2009; Borodin and Bufetov, 2021)
- New shift/flip invariance by Borodin, Gorin, and Wheeler, 2019; Galashin, 2020, from the colored stochastic 6-vertex model
Some recent developments on integrable models

Height function in the colored stochastic 6-vertex model (figure from Vadim).

\[ H^i(x, y) : \text{number of paths with color } \geq i \text{ to the right of/below } (x, y). \]
Some recent developments on integrable models

Height function in the colored stochastic 6-vertex model (figure from Vadim).

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**Theorem (Borodin, Gorin, and Wheeler, 2019)**

Let \( 1 \leq \tau \leq n, \) and \( k'_i = k_i + 1[i = \tau], U'_i = U_i + (0, 1[i = \tau]). \)
Under intersection conditions, we have

\[ \left\{ \mathcal{H}^{k_i}(U_i) \right\}_{i=1}^n \overset{d}{=} \left\{ \mathcal{H}^{k'_i}(U'_i) \right\}_{i=1}^n. \]

Passage times in colored TASEP:

\[ T^A_{B,C} = \inf \{ t \geq 0 : |\{ x \geq A + B + 1 - C : \zeta_t(x) \leq A\}| \geq C \}. \]

Corresponds to: LPP time \( L_{(1,1),(B,C)} \).
(recall: \( \{ x : \zeta_t(x) \leq A \} \) gives step initial TASEP)
New shift-invariance for colored TASEP

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One can degenerate the results in Galashin, 2020 to the following:

**Theorem**

Let \( 1 \leq \tau \leq n \) and \( A_i^+ = A_i + 1[i > \tau] \). Under intersection conditions,

\[
\max_i T_{B_i,C_i}^{A_i} \overset{d}{=} \max_i T_{B_i,C_i}^{A_i^+}.
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New shift-invariance for colored TASEP

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We get a stronger result for this.

**Theorem (Zhang, 2021)**

Let \( 1 \leq \tau \leq g \) and \( A_{i,j}^+ = A_{i,j} + 1[i > \tau] \). Under intersection conditions,
\[ \left\{ \max_{1 \leq j \leq k_i} T_{B_{i,j},C_{i,j}}^{A_{i,j}} \right\}_{i=1}^g \overset{d}{=} \left\{ \max_{1 \leq j \leq k_i} T_{B_{i,j},C_{i,j}}^{A_{i,j}^+} \right\}_{i=1}^g. \]

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The intersection conditions:

$$A_{i,j} \leq A_{i',j'}, \quad A_{i,j}^+ + B_{i,j} \geq A_{i',j'}^+ + B_{i',j'}, \quad A_{i,j}^+ - C_{i,j} \geq A_{i',j'}^+ - C_{i',j'},$$

for any $1 \leq i < i' \leq g$ and $1 \leq j \leq k_i, 1 \leq j' \leq k_{i'}$. 
New shift-invariance for colored TASEP

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$$\left\{ \max_{1 \leq j \leq k_i} T_{B_{i,j},C_{i,j}}^{A_{i,j}} \right\}_{i=1}^g \overset{d}{=} \left\{ \max_{1 \leq j \leq k_i} T_{B_{i,j},C_{i,j}}^{A_{i,j}} \right\}_{i=1}^g .$$

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for any $1 \leq i < i' \leq g$ and $1 \leq j \leq k_i, 1 \leq j' \leq k_{i'}$.

For example: by using it repeatedly, for each $N$ we have

$$\left\{ T_{N-k,k}^1 \right\}_{k=1}^{N-1} \overset{d}{=} \left\{ T_{N-k,k}^k \right\}_{k=1}^{N-1} .$$

Previously, only know that the maximum are equal in distribution.
The Oriented Swap Process
A shortest path in the group $S_N$, from $(1, \cdots, N)$ to $(N, \cdots, 1)$, swapping two neighboring numbers at a time.

$$\frac{N(N-1)}{2}$$ steps, swap $i, j$ to $j, i$ if $i < j$. 
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1. Uniform measure
2. Oriented Swap Process: Markovian according to Poisson Clocks (Angel, Holroyd, and Romik, 2009).
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A simulation with $N = 1000$ (from Angel, Holroyd, and Romik, 2009).
OSP can be viewed as the colored TASEP on an interval $[1, N]$. In Angel, Holroyd, and Romik, 2009, some truncation operators are used to connect TASEP on $\mathbb{Z}$ with TASEP on an interval.
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In particular: single particle trajectory; the finishing time of a single particle has fluctuation of $\sim N^{1/3}$ with GUE Tracy-Widom limit.
OSP and colored TASEP

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In particular: single particle trajectory; the finishing time of a single particle has fluctuation of $\sim N^{1/3}$ with GUE Tracy-Widom limit.

**Absorbing time**: the time when the OSP terminates.

**Question**

*What are the fluctuations and limiting law of the absorbing time?*
A conjecture on the finishing times

Take \( U_N = (U_N(1), \ldots, U_N(N-1)) \), where \( U_N(k) \) is the last time such that a swap happens between the sites \( k \) and \( k + 1 \).

**Conjecture (Bisi, Cunden, Gibbons, and Romik, 2020; Bufetov, Gorin, and Romik, 2020)**

\[
U_N \overset{d}{=} \{ L_{(1,1), (k,N-k)} \}_{k=1}^{N-1}.
\]
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Take $\mathbf{U}_N = (U_N(1), \ldots, U_N(N - 1))$, where $U_N(k)$ is the last time such that a swap happens between the sites $k$ and $k + 1$.

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$$\mathbf{U}_N \overset{d}{=} \{L_{(1,1),(k,N-k)}\}_{k=1}^{N-1}.$$

Some results

1. Single $k$.
2. $N \leq 6$ (computer-assisted).
3. $\max_{1 \leq k \leq N-1} U_N(k) \overset{d}{=} \max_{1 \leq k \leq N-1} L_{(1,1),(k,N-k)}$
A conjecture on the finishing times

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Some results

1. Single $k$.
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3. $\max_{1 \leq k \leq N-1} U_N(k) \overset{d}{=} \max_{1 \leq k \leq N-1} L_{(1,1),(k,N-k)} \Rightarrow$ OSP absorbing time converges to GOE Tracy-Widom.
Result on OSP and implications

Theorem (Zhang, 2021)

\[ U_N \overset{d}{=} \left\{ L_{(1,1),(k,N-k)} \right\}_{k=1}^{N-1} . \]
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\[ U_N \overset{d}{=} \left\{ L_{(1,1), (k,N-k)} \right\}_{k=1}^{N-1}. \]

Some implications (using the asymptotic results of LPP):

1. Under \( N^{2/3}, N^{1/3} \) scaling, \( U_N \) converges to the parabolic Airy_2 process.

2. Consider \( k_* \) such that the last swap is between sites \( k_* \) and \( k_* + 1 \); then \( N^{-2/3} (k_* - N/2) \) converges.

3. In scale smaller than \( N^{2/3} \), \( U_N \) converges to simple random walk.
Proof ideas
From the colored TASEP shift invariance to OSP finishing times:

\[
\left\{ L_{(1,1),(k,N-k)} \right\}_{k=1}^{N-1} \overset{d}{=} \left\{ T_{N-k,k}^{1} \right\}_{k=1}^{N-1} \overset{d}{=} \left\{ T_{N-k,k}^{k} \right\}_{k=1}^{N-1} \overset{d}{=} U_N.
\]

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\]


Shift invariance: an example

Take \(B, C \geq 2\). Goal: show that \(T_{B,1}^0, T_{1,C}^0 \overset{d}{=} T_{B,1}^0, T_{1,C}^1\).

\(T_{B,1}^0, T_{1,C}^0\): TASEP with labels \(\leq 0\)
From colored TASEP shift invariance to OSP finishing times:

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**Shift invariance: an example**

Take \(B, C \geq 2\). Goal: show that \(T^0_{B,1}, T^0_{1,C} \overset{d}{=} T^0_{B,1}, T^1_{1,C}\).

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From colored TASEP shift invariance to OSP finishing times:

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\( T^0_{B,1}, T^1_{1,C} \):

Since time \( T^0_{2,1} \), the blue particle is to the right of the red particle \( \Rightarrow \) independent evolution.
From colored TASEP shift invariance to OSP finishing times:

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\{ L_{(1,1),(k,N-k)} \}_{k=1}^{N-1} \overset{d}{=} \{ T^1_{N-k,k} \}_{k=1}^{N-1} \overset{d}{=} \{ T^k_{N-k,k} \}_{k=1}^{N-1} \overset{d}{=} U_N.
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**Shift invariance: an example**

Take \( B, C \geq 2 \). Goal: show that \( T^0_B, T^0_1 \overset{d}{=} T^0_B, T^1_1 \).

\( T^0_B, T^0_1 \): TASEP with labels \( \leq 0 \)

Since time \( T^0_2 \), the blue particle is to the right of the red particle
\( \Rightarrow \) independent evolution.

Need ‘equal’ in distribution of the configurations at \( T^0_2 \);

Use \( \max\{ T^0_{B',1}, T^0_{1,C'} \} \overset{d}{=} \max\{ T^0_{B',1}, T^1_{1,C'} \} \).
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Since time \( T^0_{2,1} \), the blue particle is to the right of the red particle \( \Rightarrow \) independent evolution.

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**General: inductive arguments**
Further questions
Can some of the constraints be relaxed?

For \( P[T_{B_1,C_1}^{A_1} < t_1, T_{B_2,C_2}^{A_2} < t_2] = P[T_{B_1,C_1}^{A_1} < t_1, T_{B_2,C_2}^{A_2} < t_2] \), need

1. \( A_1 \leq A_2, A_2' \)
2. \( A_1 - C_1 \geq A_2 - C_2, A_2' - C_2 \)
3. \( A_1 + B_1 \geq A_2 + B_2, A_2' + B_2 \)
Can some of the constraints be relaxed?

For $\mathbb{P}[T_{B_1,C_1}^{A_1} < t_1, T_{B_2,C_2}^{A_2} < t_2] = \mathbb{P}[T_{B_1,C_1}^{A_1} < t_1, T_{B_2,C_2}^{A_2'} < t_2]$, need

1. $A_1 \leq A_2, A_2'$
2. $A_1 - C_1 \geq A_2 - C_2, A_2' - C_2$
3. $A_1 + B_1 \geq A_2 + B_2, A_2' + B_2$

For $t_1 = t_2$, just need (1) and

4. $A_1 + B_1 - C_1 \geq A_2 + B_2 - C_2, A_2' + B_2 - C_2$

Note that (1)+(2)+(3) implies (1)+(4).
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Question: what is the key property? Crossing of paths?
Can some of the constraints be relaxed?

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Scaling limit of the colored TASEP?

Two families of TASEPs: LPP and colored TASEP.

LPP $\rightarrow$ Airy Sheet

$(x, y) \mapsto n^{-1/3}(L_{(xn^{2/3}, -xn^{2/3}), (n-yn^{2/3}, n+yn^{2/3})} - 4n)$

Colored TASEP?

$(x, y) \mapsto n^{-1/3}(T_{n^{2/3}(y-x), n-2/3(y-x)}^{n^{2/3}(y-x), n-2/3(y-x)} - 4n)$?
Thank you!


