A cutoff transition for repeated averages

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Consider a following Markov chain on $\mathbb{R}^n$ (Bourgain '80):

Start with $x_0 = (x_{0,1}, \ldots, x_{0,n}) \in \mathbb{R}^n$. At step k, given x_k,

1. pick two distinct coordinates I and J uniformly at random,
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Related studies/models: [Chatterjee, Seneta, 77] [Feller, 68]; consensus algorithm [Olshevsky, Tsitsiklis, 09] [Shah, 08]; local iterated averaging [Diaconis, Saloff-Coste, 12]; convergence on general graphs [Aldous, Lanoue, 12]; Deffuant model [Häggström, 12] [Lanchier, 12]; Kac walk [Kac, 54] [Pillai, Smith, 17]

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In terms of L^2: can be explicitly computed

$$\mathbb{E}\left(\sum_{i=1}^{n}(x_{k,i} - \bar{x}_0)^2\right) = \left(1 - \frac{1}{n-1}\right)^k \sum_{i=1}^{n}(x_{0,i} - \bar{x}_0)^2$$

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More relevant (and difficult): L^1 distance to $(\bar{x}_0, \dots, \bar{x}_0)$?

Consider $T(k) = \sum_{i=1}^n |x_{k,i} - \bar{x}_0|$.

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Consider $T(k) = \sum_{i=1}^{n} |x_{k,i} - \bar{x}_0|$.

The initial condition matters:
take $x_0 = (1, 0, \ldots, 0)$, a worst case by linearity.

Take $x_0 = (1, 0, \ldots, 0)$, and consider $T(k) = \sum_{i=1}^{n} |x_{k,i} - \bar{x}_0|$.

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How it decays between $\frac{n}{2} \log n$ and $n \log n$?

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Theorem (Chatterjee-Diaconis-Sly-Z. '20)

As $n \to \infty$, we have in probability convergence

$$T(\theta n \log n) \to 2, \qquad \text{for any } \theta < (2 \log 2)^{-1},$$

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This also has a Gaussian cutoff profile.

Theorem (Chatterjee-Diaconis-Sly-Z. '20)

Let $\Phi : \mathbb{R} \to [0, 1]$ be the cumulative distribution function of the standard normal distribution. For any $a \in \mathbb{R}$, as $n \to \infty$ we have

$$T(\lfloor n(\log_2(n) + a\sqrt{\log_2(n)})/2 \rfloor) \to 2\Phi(-a)$$

in probability .

Proof Ideas

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W.h.p., initially be like

$$1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,$$

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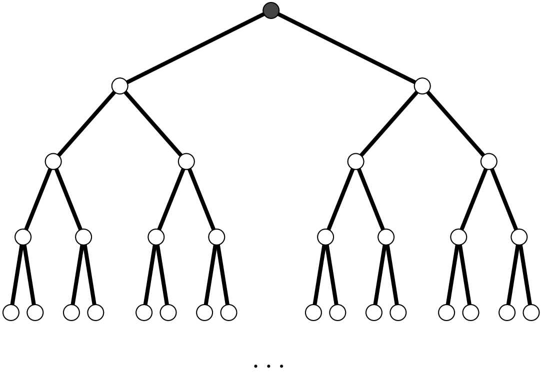
$$\frac{3}{8}, 0, 0, 0, 0, \frac{3}{8}, 0, 0, 0, \frac{1}{4}, 0, 0, 0, 0, 0,$$

Unlikely to happen.

Proof Ideas

Tree structure:



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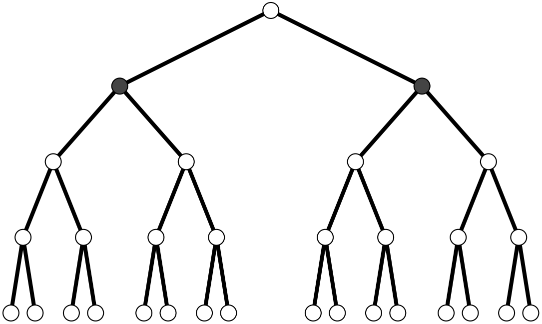

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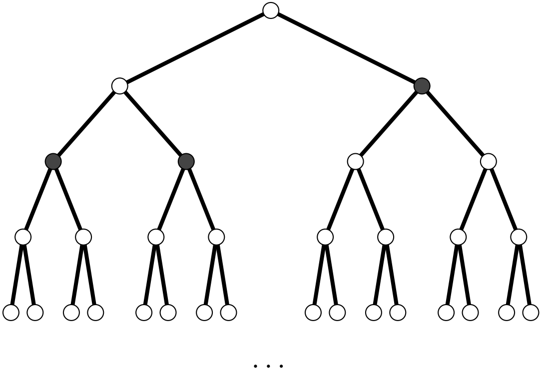

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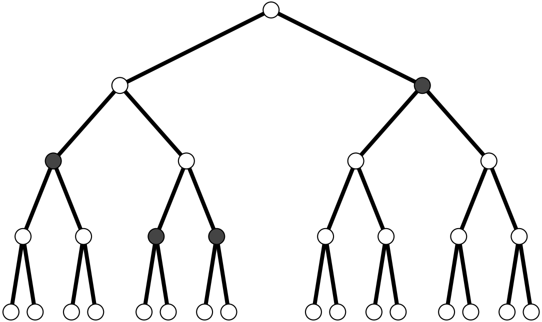

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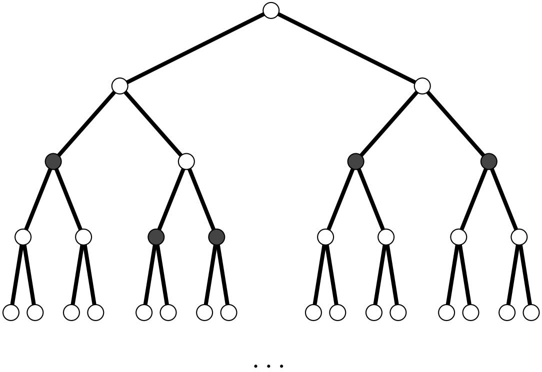

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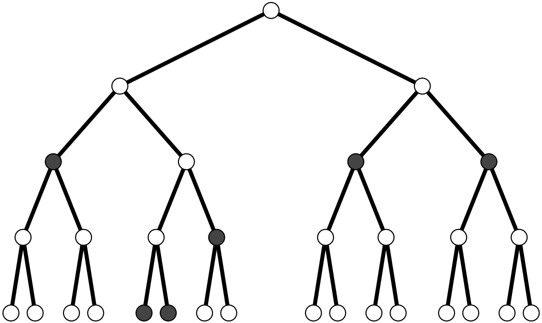

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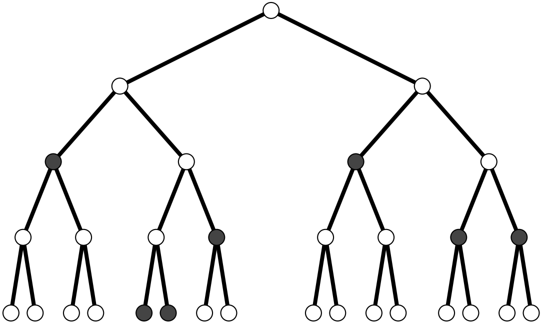

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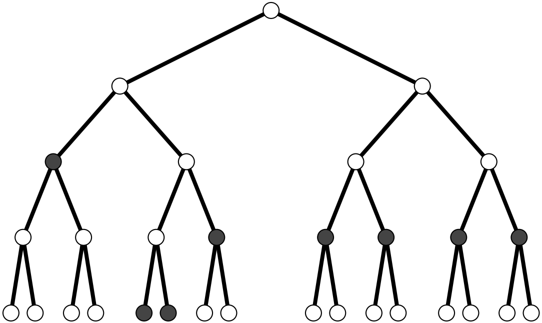

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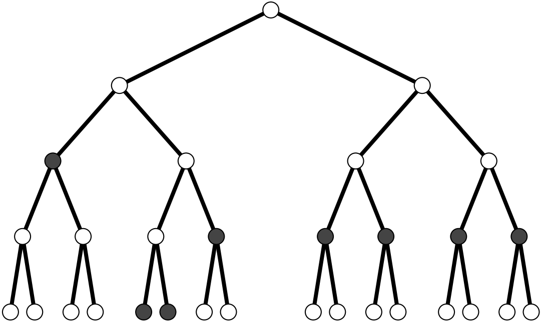

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Each particle at level i corresponds to one 2^{-i}.

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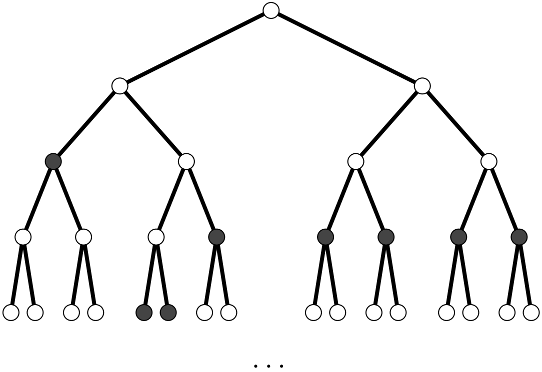

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Critical level $\log_2(n)$: a particle at level $\log_2(n)$ is reached at time $\frac{n}{2}(\log_2(n) + \sqrt{\log_2(n)}\mathcal{N}(0,1))$.

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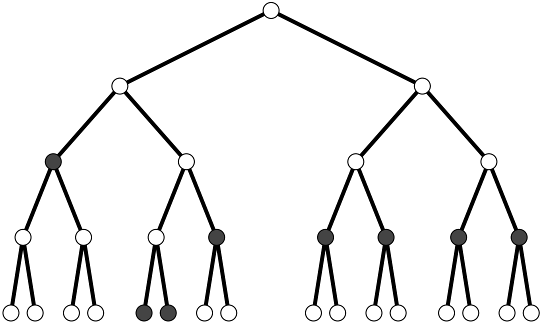

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$\Rightarrow$ For $k < \frac{n}{2}(\log_2(n) - C\sqrt{\log_2(n)})$: most coordinates are 0.

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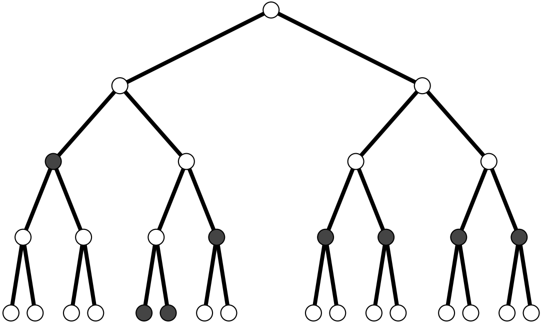

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$\Rightarrow$ For $k < \frac{n}{2}(\log_2(n) - C\sqrt{\log_2(n)})$: most coordinates are 0.

1. $T(k) = 2 - o(1)$.
2. This tree is a good approximation.

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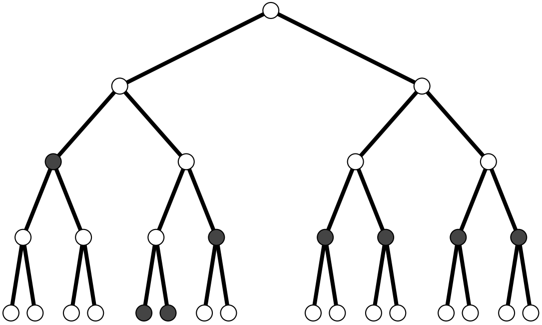

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For $k \approx \frac{n}{2} \log_2(n)$: most weights are $O(\frac{1}{n})$.

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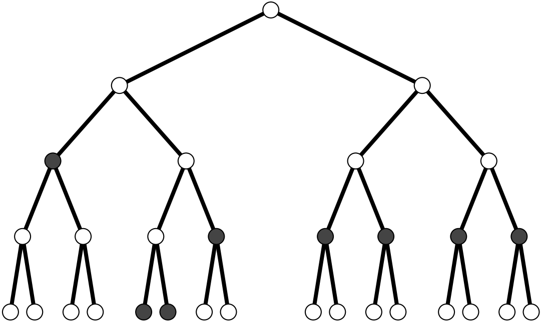

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For $k \approx \frac{n}{2}\log_2(n)$: most weights are $O(\frac{1}{n})$.

$\Rightarrow$ L^2-distance is of order $O(n^{-1})$; run for Cn more steps to get $o(n^{-1})$, then by Cauchy-Schwarz $T(k) = o(1)$.

Thank you!

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