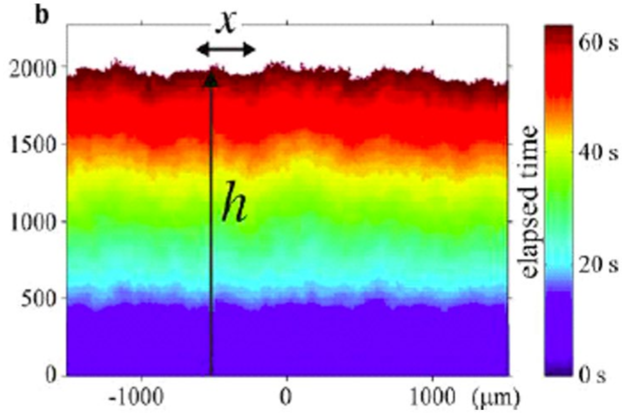
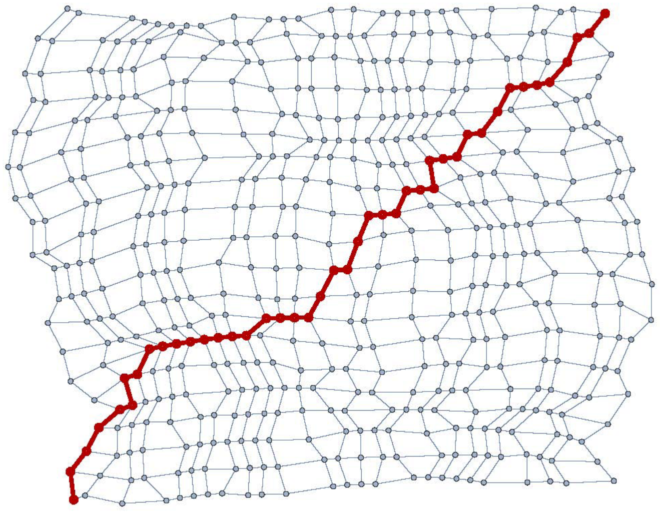


From the KPZ fixed point to the directed landscape



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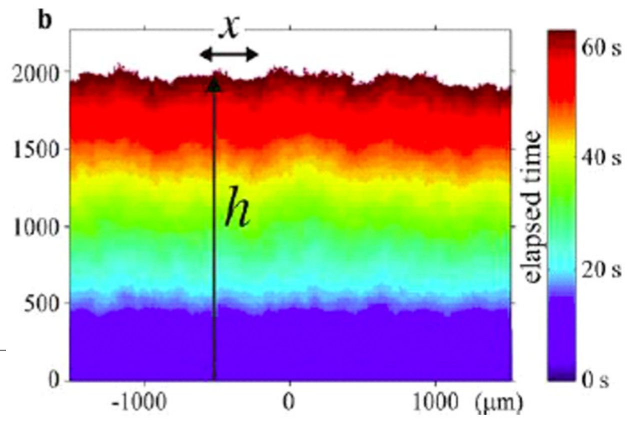
Seminário de Probabilidade e Mecânica Estatística **Feb 12, 2025**

Joint work with Duncan Dauvergne (arxiv:2412.13032)



KPZ fixed point (KPZ FP)

Markov process in the UC space

Space of all upper semi-continuous functions $f:\mathbb{R}\to\mathbb{R}\cup\{-\infty\}$, with $\sup\frac{f(x)}{1+|x|}<\infty$

Transition probability explicit:

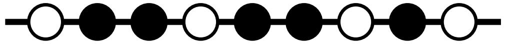
(starting from h_0) $P_{h_0}(h(t,x_i)\le r_i,\ i=1,\ldots,m)=\det(I-\mathbf{K}_{h_0,x_i,r_i,t})_{L^2(\mathbb{R}_+;\mathbb{R}^m)}$

where $(\mathbf{K}_{h_0,x_i,r_i,t})_{ij}=\lim_{L\to\infty} e^{-\frac{1}{3}t\partial^3-(x_i+L)\partial^2+r_i\partial}(\mathbf{P}^{\text{Hit }h_0}_{-L,L}-\mathbf{1}_{x_i<x_j})e^{\frac{1}{3}t\partial^3+(x_j-L)\partial^2-r_j\partial}$

Scaling limit of 1D exclusion processes/growth model, KPZ equation, etc.

KPZ fixed point (KPZ FP)

1D exclusion process

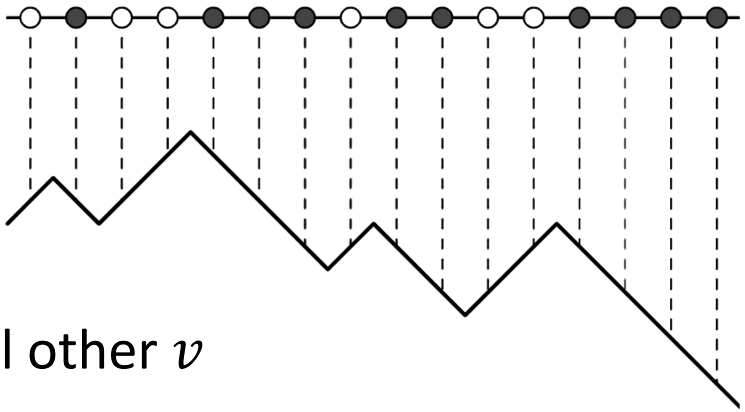


- Particle at x jumps to $x+v$ with rate $p(v)$, if $x+v$ is empty
- Jump generator $p: \mathbb{Z} \to \mathbb{R}_{\geq 0}$, such that $\{v: p(v)+p(-v)>0\}$ is finite and generates $\mathbb{Z}$
- (normalized) asymmetric assumption: $\sum_v vp(v) = 1$

Height function $h: \mathbb{Z} \to \mathbb{Z}$, such that $h(x+1) = h(x)+1$ if x occupied, $h(x+1) = h(x)-1$ if x empty

1:2:3 scaling $h_\varepsilon(t,x) = \varepsilon^{1/2} h(2\varepsilon^{-3/2}t, 2\varepsilon^{-1}x) - \varepsilon^{-1}t$

$h_\varepsilon \to$ KPZ fixed point as $\varepsilon \to 0$



(Matetski-Quastel-Remenik, 16')
Totally asymmetric nearest neighbor (TASEP): $p(1)=1$, $p(v)=0$ all other v

(Quastel-Sarkar, 20') General p; slightly weaker sense for non-nearest neighbor

KPZ fixed point (KPZ FP)

KPZ equation *(Kardar-Parisi-Zhang, 86')*

$$\partial_t h = \frac{1}{4}(\partial_x h)^2 + \frac{1}{4}\partial_x^2 h + \xi$$

Long time limit: 1:2:3 scaling $h \mapsto \delta h(\delta^{-3}t, \delta^{-2}x)$

$$\partial_t h = \frac{1}{4}(\partial_x h)^2 + \frac{\delta}{4}\partial_x^2 h + \delta^{1/2}\xi$$

As $\delta \to 0$, KPZ equation converges to KPZ FP *(Quastel-Sarkar, 20')*; also *(Virag, 20')*

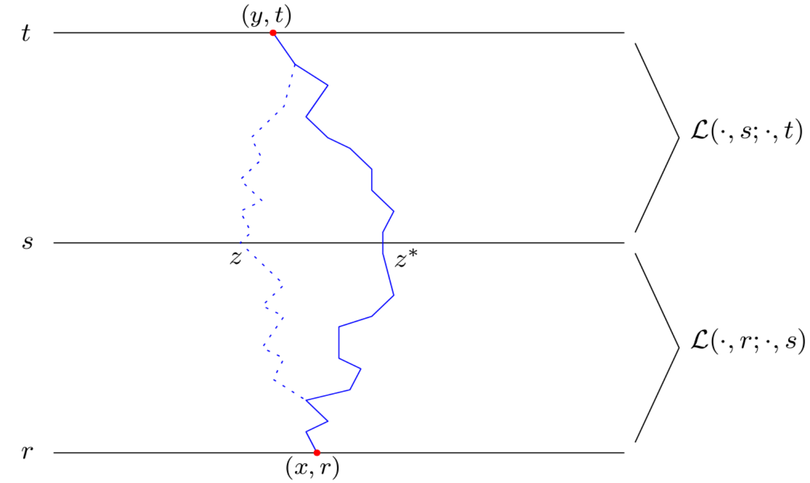
Let's next introduce the other object, the **directed landscape**

Directed landscape (DL)

A four-parameter random function $\mathcal{L} : \mathbb{R}^4_\uparrow \to \mathbb{R}$ $\quad \mathbb{R}^4_\uparrow = \{(x, s; y, t) \in \mathbb{R}^4 : s < t\}$

Directed metric, $\mathcal{L}(x, s; y, t)$ is the distance from (x, s) to (y, t)

$$\mathcal{L}(x, r; y, t) = \max_{z \in \mathbb{R}} \mathcal{L}(x, r; z, s) + \mathcal{L}(z, s; y, t)$$

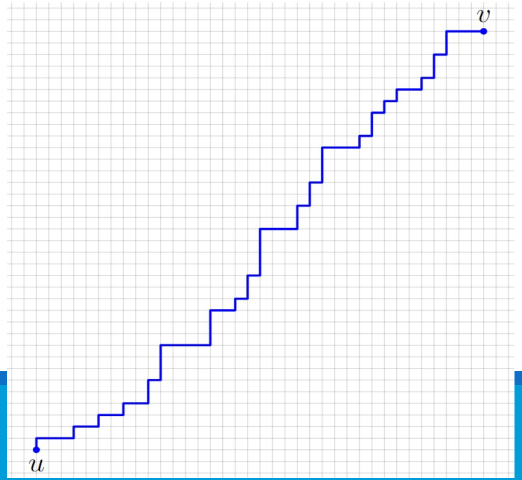


Example Exponential Last Passage Percolation (LPP)

- $\xi(v) \sim \mathrm{Exp}(1)$, $\forall v \in \mathbb{Z}^2$ independently
- Passage time: $T(u, v) \coloneqq \max_{\gamma} \sum_{w \in \gamma} \xi(w)$, over all up-right path γ

$2^{-\frac{4}{3}} n^{-\frac{1}{3}} \left(T(R_n(x, s), R_n(y, t)) - 4n(t - s) - 2^{\frac{8}{3}} n^{\frac{2}{3}} (y - x) \right)$ converges to $\mathcal{L}$ as $n \to \infty$

$$R_n : (x, s) \mapsto (ns + 2^{5/3} n^{2/3} x, ns)$$



(Dauvergne-Ortmann-Virag, 18') Brownian LPP

(Dauvergne-Virag, 21') Other LPPs

Directed landscape (DL)

A four-parameter random function $\mathcal{L} : \mathbb{R}^4_\uparrow \to \mathbb{R}$ $\mathbb{R}^4_\uparrow = \{(x, s; y, t) \in \mathbb{R}^4 : s < t\}$

Some natural properties

For disjoint $\{(t_i, s_i) : i \in \{1, \dots k\}\}$ $\mathcal{L}(\cdot, t_i; \cdot, s_i)$, $i \in \{1, \dots, k\}$ are independent

Symmetry: 1. *(Time stationarity)*

$$\mathcal{L}(x, t; y, t+s) \stackrel{d}{=} \mathcal{L}(x, t+r; y, t+s+r).$$

2. *(Spatial stationarity)*

$$\mathcal{L}(x, t; y, t+s) \stackrel{d}{=} \mathcal{L}(x+c, t; y+c, t+s).$$

3. *(Flip symmetry)*

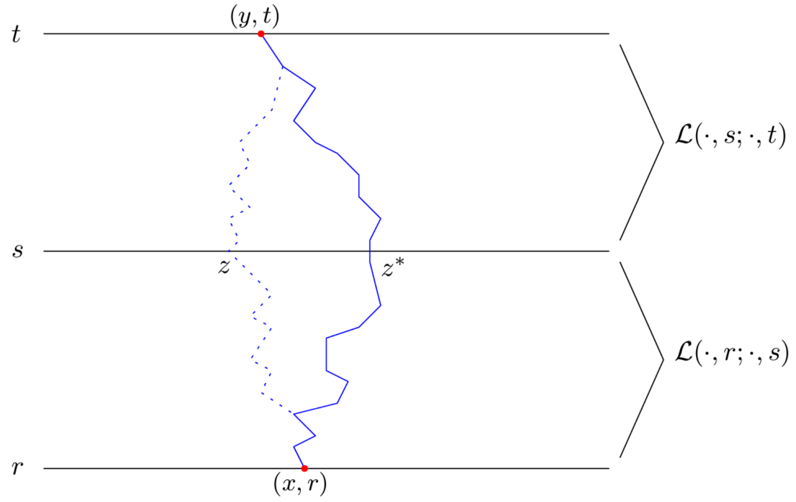
$$\mathcal{L}(x, t; y, t+s) \stackrel{d}{=} \mathcal{L}(-y, -s-t; -x, -t).$$

4. *(Skew stationarity)*

$$\mathcal{L}(x, t; y, t+s) \stackrel{d}{=} \mathcal{L}(x+ct, t; y+ct+sc, t+s) + s^{-1}[(x-y-sc)^2 - (x-y)^2].$$

5. *(Rescaling)*

$$\mathcal{L}(x, t; y, t+s) \stackrel{d}{=} q\mathcal{L}(q^{-2}x, q^{-3}t; q^{-2}y, q^{-3}(t+s)).$$



Relation between KPZ FP and DL

DL generate KPZ FP;   KPZ FP are marginals of DL

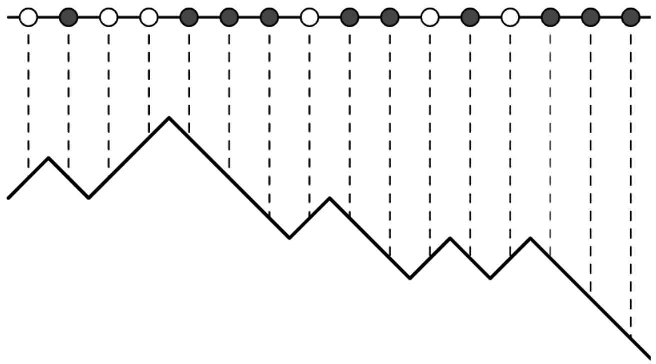
$$\mathcal{L} : \mathbb{R}^4_\uparrow \to \mathbb{R} \qquad \mathbb{R}^4_\uparrow = \{(x, s; y, t) \in \mathbb{R}^4 : s < t\}$$

Example $\mathcal{L}(0,0; \cdot\,, t)$ = KPZ FP from δ_0 at time t

$\delta_0(0) = 0$,
$\delta_0(x) = -\infty$ for $x \neq 0$

Variational formula KPZ FP starting from h_0 is given by $h_t(y) = \sup_{x\in\mathbb{R}}(h_0(x) + \mathcal{L}(x, 0; y, t))$

Why? A coupling between TASEP and Exponential LPP



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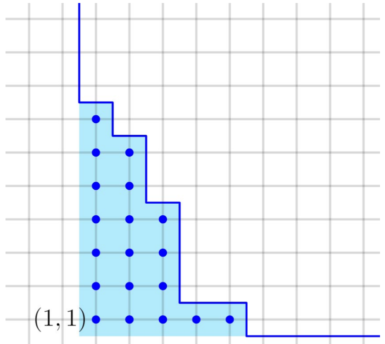


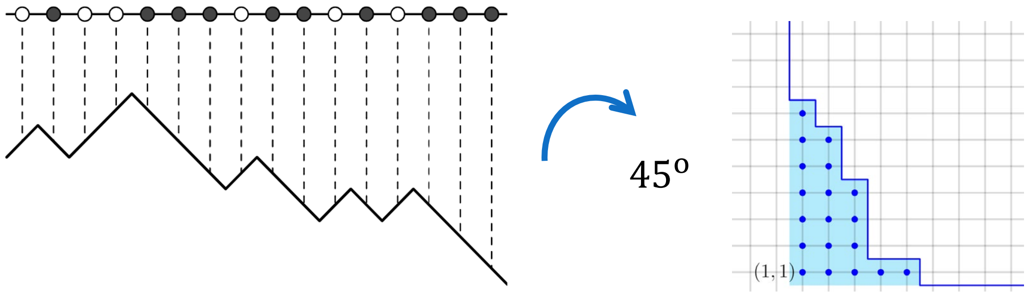
Relation between KPZ FP and DL

Variational formula: KPZ FP starting from h_0 is $h_t(y) = \sup_{x \in \mathbb{R}} (h_0(x) + \mathcal{L}(x, 0; y, t))$

DL = a coupling of multiple KPZ FP

(same dynamics, different initial data)

For TASEP $(p(1) = 1, p(v) = 0$ all other $v)$



TASEP → **KPZ FP**

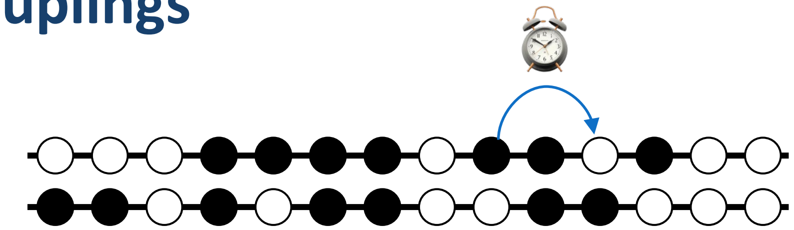
Exponential LPP → **DL**

DL seems to contain more information than KPZ FP?

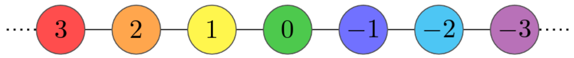
Is DL canonical for KPZ FP? Or is it special and just for LPP?

Other natural couplings

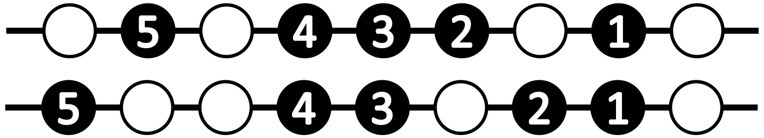
❖ **Basic coupling**



(For ASEP, i.e., $p(v) = 0$ all $v \neq \pm 1$, also called colored ASEP)



❖ **Particle coupling**



Synchronize Poisson clocks on particles

Our result: unify KPZ FP and DL

DL generates KPZ FP $h_t(y) = \sup_{x\in\mathbb{R}}(h_0(x) + \mathcal{L}(x,0;y,t))$

DL seems to contain more information than KPZ FP?

Is DL canonical for KPZ FP? Or is it special and just for LPP?

(Dauvergne-Z., 24) For a family of random operators $\{\mathcal{K}_{s,t}\}_{s<t}$ on the UC space, if

1. KPZ fixed point $\mathcal{K}_{s,t}f$ has the same law as KPZ FP from f running for time $t-s$
2. Independent increments For any disjoint intervals $\{(s_i,t_i)\}_{i=1}^k$, $\mathcal{K}_{s_i,t_i}$ are independent
3. Monotonicity $\mathcal{K}_{r,t}g \leq \mathcal{K}_{s,t}f$, for any $s \leq r < t$ and $g \leq \mathcal{K}_{s,r}f$
4. Shift-invariant $\mathcal{K}_{s,t}(f+c) = \mathcal{K}_{s,t}f + c$

Then $\{\mathcal{K}_{s,t}\}_{s<t}$ can be coupled with $\mathcal{L}$, such that $\mathcal{K}_{s,t}f = \sup_x f(x) + \mathcal{L}(x,s;\cdot,t)$.

Implication 1. KPZ FP contains all information of DL

2. All natural couplings should converge to DL (verify 2,3,4)

Applications to DL convergence

(Dauvergne-Z., 24)

1. **ASEPs** under various couplings, e.g., basic/particle couplings
   (The basic coupling case, i.e., colored ASEP, proved in Aggarwal-Corwin-Hegde, 24')
2. **General 1D exclusion process** under basic coupling
   (1 & 2 use KPZ FP convergence in Quastel-Sarkar, 20')
3. **KPZ equations** coupled with the same noise
   (Using KPZ FP convergence in Quastel-Sarkar, 20' or Virag 20'; recover Wu, 23')
4. **O'Connell-Yor polymer** (KPZ FP convergence in Virag 20')
5. **Brownian web/Coalescing random walk distance** (KPZ FP convergence in Veto-Virag, 23')
6. ***Variants of TASEP: PushASEP, inhomogeneous speed, etc.*** *(KPZ FP convergence in Matetski-Remenik 23')*

In summary: we provide a framework to upgrade KPZ FP convergence to DL convergence

Proof ideas

Core task upgrade KPZ FP marginals to DL

(Dauvergne-Z., 24) A characterization/uniqueness of DL:

Suppose that $\mathcal{M}: \mathbb{R}^4_\uparrow \to \mathbb{R}$, $\mathbb{R}^4_\uparrow = \{(x,s;y,t) \in \mathbb{R}^4 : s < t\}$, is continuous, and

1. For any $r < s < t$, and x, y, z, we have $\mathcal{M}(x,r;y,t) \geq \mathcal{M}(x,r;z,s) + \mathcal{M}(z,s;y,t)$
2. For any disjoint intervals $\{(s_i,t_i)\}_{i=1}^k$, $\mathcal{M}(\cdot, s_i; \cdot, t_i)$ are independent
3. For any $s < t$ and UC f, g supported on finitely many points, we have (in distribution)

$$\sup_{x,y} \mathcal{M}(x,s;y,t) + f(x) + g(y) = \sup_{x,y} \mathcal{L}(x,s;y,t) + f(x) + g(y)$$

Then $\mathcal{M}$ must be the directed landscape, i.e., has the same law as $\mathcal{L}$.

TODO the joint law of $\mathcal{M}(x_1,0;y_1,1), \mathcal{M}(x_2,0;y_2,1)$ and $\mathcal{L}(x_1,0;y_1,1), \mathcal{L}(x_2,0;y_2,1)$ are the same

Using a *Lindeberg exchange strategy,* plus careful quantitative analysis

(Originated in Lindeberg's proof of CLT, 1922; widely used in e.g., hydrodynamics in Bohadoran-Guiol-Ravishankar-Saada, 02'; spin glass in Chatterjee, 04'; various problems in Mossel-O'Donnell-Oleszkiewicz 05'; random matrices: Chatterjee, 05', Tao-Vu, 07', Knowles-Yin, 17'; 2D polymer/stochastic heat flow, Caravenna-Sun-Zygouras, 21', Tsai, 24')

Summary

- ❖ We show that, under natural assumptions
(independent increments, monotonicity, and shift commutativity)
the directed landscape is the **only** object with KPZ fixed point marginals
  i.e., DL is intrinsic to KPZ FP

- ❖ This gives a **framework** of upgrading KPZ fixed point convergence to directed landscape convergence

- ❖ Effectively, we give an **alternative construction** of the directed landscape

- ❖ Our arguments are **robust**, and can potentially be adapted to other settings (open boundary? periodic?)