The Environment Seen from Geodesics in Exponential Last Passage Percolation

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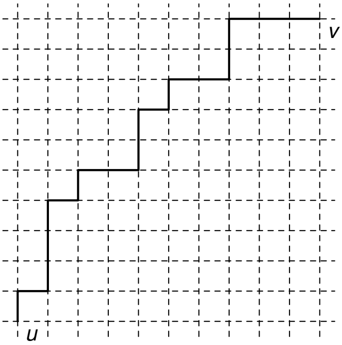
The model: directed LPP with exponential weights



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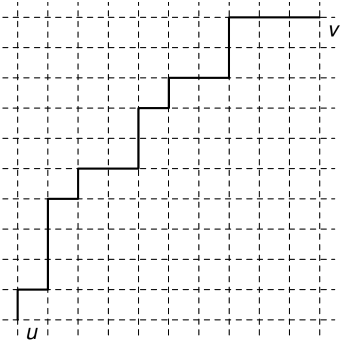
- $\xi(v) \sim \text{Exp}(1)$, i.i.d. $\forall v \in \mathbb{Z}^2$
- Passage time: $T_{u,v} := \max_\gamma \sum_{w\in\gamma} \xi(w)$
- Geodesic: $\Gamma_{u,v} := \operatorname{argmax}_\gamma \sum_{w\in\gamma} \xi(w)$



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Equivalent to TASEP, exactly solvable with $1:2:3$ scaling.

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$$2^{-4/3}n^{-1/3}\left(T_{(0,0),(n-x(2n)^{2/3},n+x(2n)^{2/3})} - 4n\right) \Rightarrow \mathcal{A}_2(x) - x^2$$

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  $\mathcal{A}_2$ is absolute continuous with respect to Brownian motion (Corwin and Hammond, 2014).
- General initial data: KPZ fixed point (Matetski, Quastel, and Remenik, 2017).
  Joint scaling limit: the directed landscape (Dauvergne, Ortmann, and Virág, 2018).

The problem and our results

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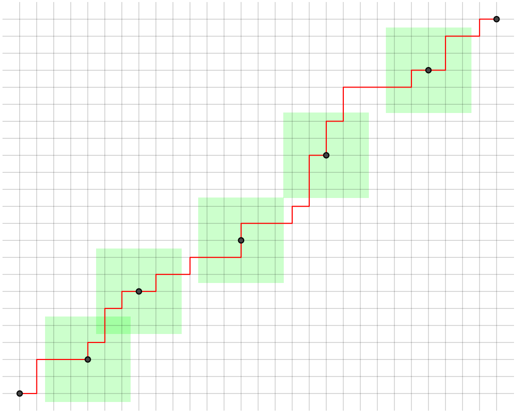

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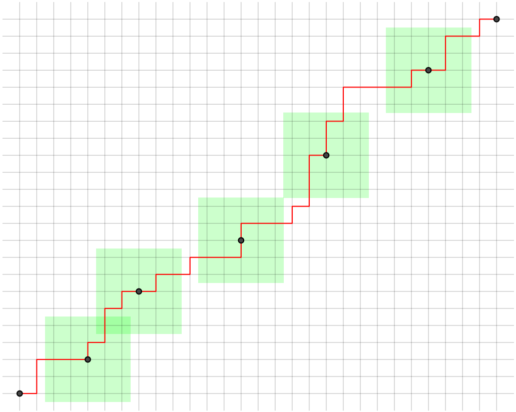

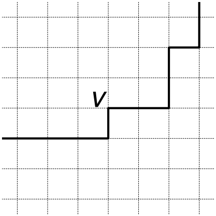

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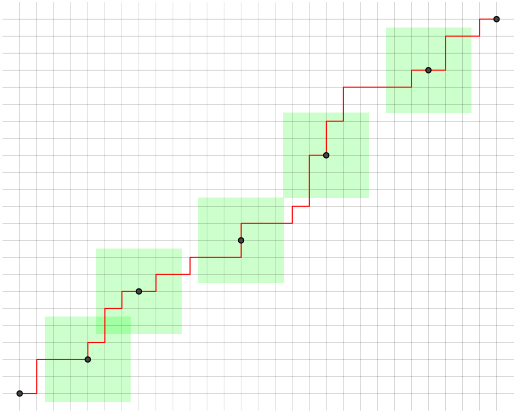

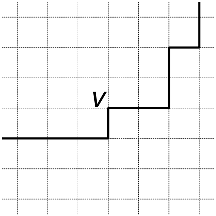

- Random environment at v contains: $\xi\{v\} = \{\xi(v+u)\}_{u\in\mathbb{Z}^2} \in \mathbb{R}^{\mathbb{Z}^2}$ and $\Gamma_{(0,0),(n,n)} - v \in \{0,1\}^{\mathbb{Z}^2}$.

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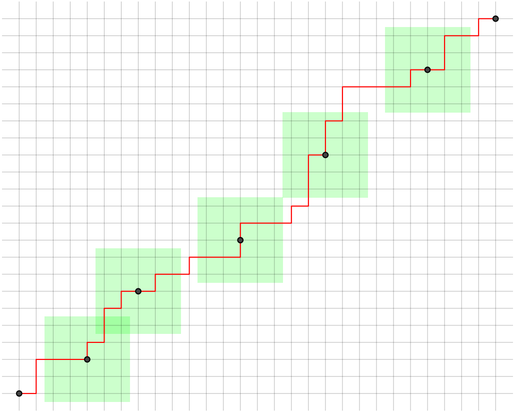

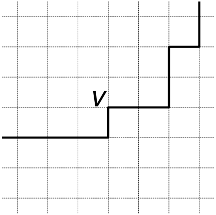

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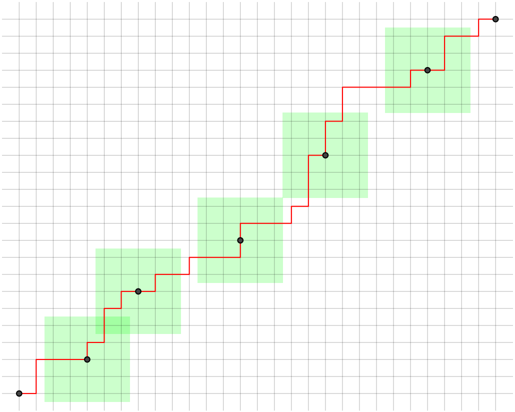

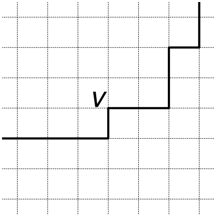

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- Consider the environment for all $v \in \Gamma_{(0,0),(n,n)}$.
- Empirical measure: $\mu_n := \frac{1}{|\Gamma_{(0,0),(n,n)}|} \sum_{v\in\Gamma_{(0,0),(n,n)}} \delta_{(\xi\{v\},\Gamma_{(0,0),(n,n)}-v)}$.

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In a different direction, (Bates, 2020) proved convergence of the empirical measure of weights $\frac{1}{|\Gamma_{(0,0),(n,n)}|} \sum_{e \in \Gamma_{(0,0),(n,n)}} \delta_{\xi(e)}$, for certain families of i.i.d. edge weights, using a variational formula.

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- Also consider $\Gamma_{(0,0)} = \{\Gamma_{(0,0)}[i]\}_{i=1}^{\infty}$, the semi-infinite geodesic in the $(1,1)$ direction; let
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Theorem (Sly and Z., unpublished)

There is a (deterministic) measure ν, such that

1. *$\mu_n \to \nu$ in probability.*
2. *The law of $\xi\{v\}, \Gamma_{(0,0),(n,n)} - v$ converges to ν, where v is the midpoint of $\Gamma_{(0,0),(n,n)}$.*
3. *$\overline{\mu}_r \to \nu$ almost surely.*
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Similar results hold for geodesics in other directions.

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$$\to P[\bar{\xi}((0,0)) > x] = (1 + \frac{3x}{4} + \frac{x^2}{8})e^{-x}.$$

(Note that before we know $\mathbb{E}\bar{\xi}((0,0)) = 2$, since $\mathbb{E}T_{(0,0),(n,n)} \sim 4n$.)

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- Portion of 'turnings' along the geodesic:

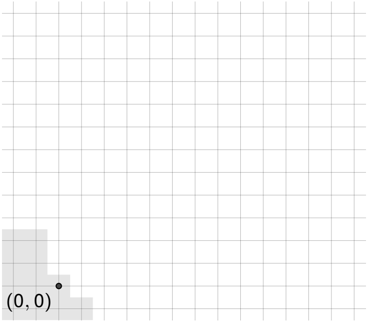
  Let N_n be the number of $v \in \mathbb{Z}^2$, such that $\{v, v-(1,0), v+(0,1)\} \subset \Gamma_{(0,0),(n,n)}$, or $\{v, v+(1,0), v-(0,1)\} \subset \Gamma_{(0,0),(n,n)}$.

  Then $\frac{N_n}{2n} \to \frac{3}{8}$ in probability.

Semi-infinite geodesic $\Leftrightarrow$ Competition interface from stationary

The limiting measure

Semi-infinite geodesic ⇔ Competition interface from stationary



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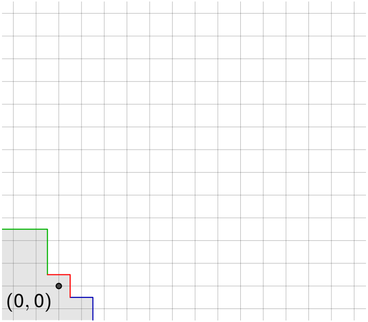

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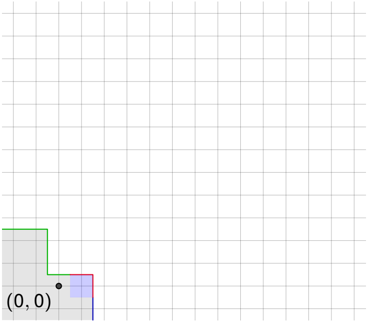

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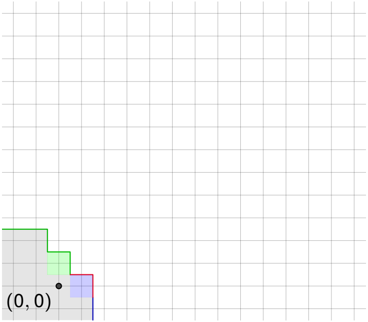

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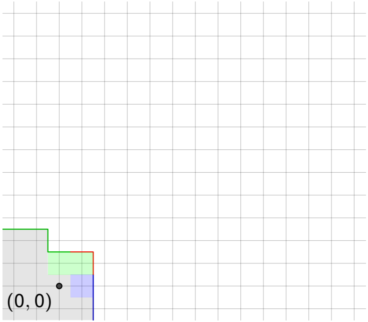

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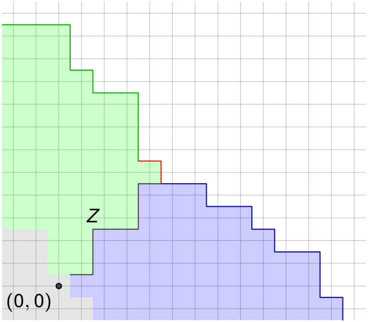

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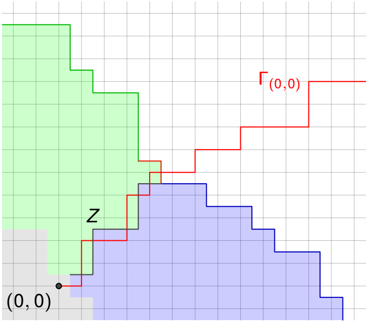

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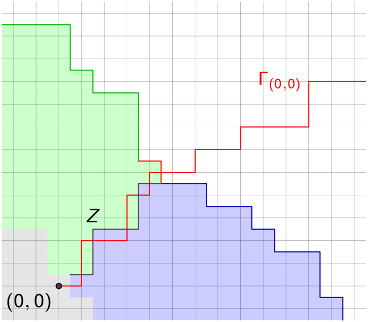

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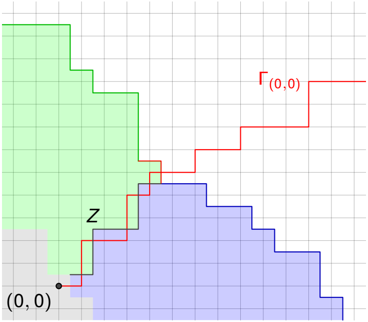

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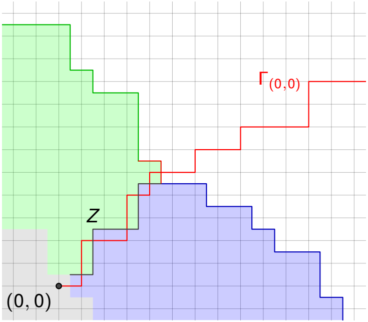

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  $\Rightarrow \xi(v) = G(v + (1,0)) \wedge G(v + (0,1)) - G(v)$.
- Boundary of $I = \{v : G(v) \le 0\}$ is a (two-sided) simple random walk. Define $\xi^{\vee}(v) = G(v) - G(v - (1,0)) \vee G(v - (0,1))$. Given I, $\{\xi^{\vee}(v)\}_{v \notin I}$ are i.i.d. $\mathrm{Exp}(1)$. (Seppäläinen, etc.)

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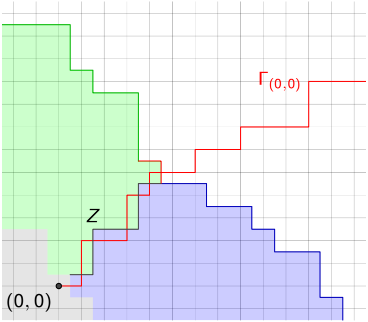

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- Let the aggregate at time t be $\{v : G(v) \leq t\}$. Then $\xi^{\vee}(v)$ is the waiting time at v, and $Z = \Gamma_{(0,0)} + (\frac{1}{2}, \frac{1}{2})$.

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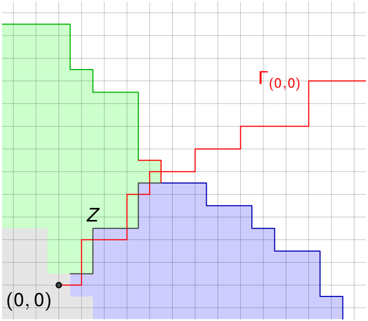

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- Let $G(v)$ be the time when v is occupied.
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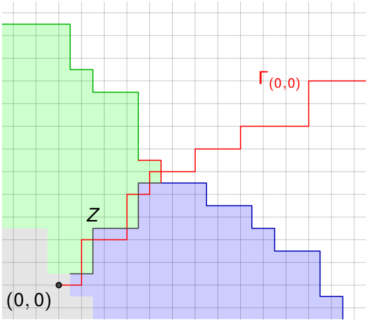

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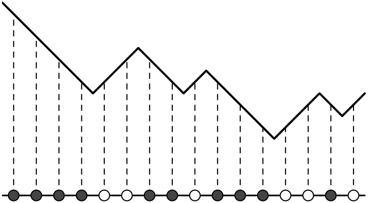
- Now suffices to study local environment around the competition interface.

Competition interface $\Leftrightarrow$ TASEP with a second class particle

Competition interface $\Leftrightarrow$ TASEP with a second class particle

- TASEP and growing surface:



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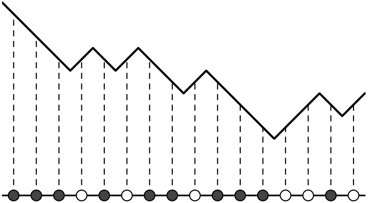

The limiting measure

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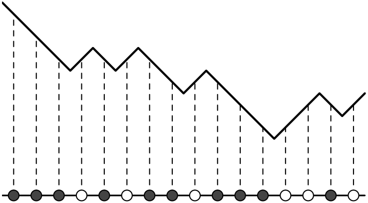

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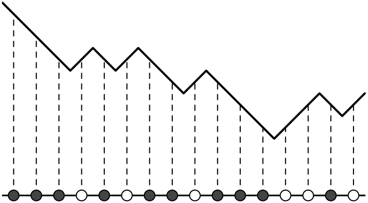

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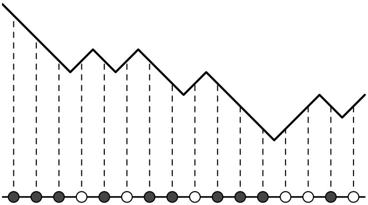

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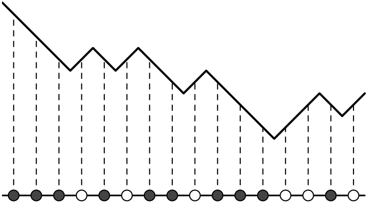

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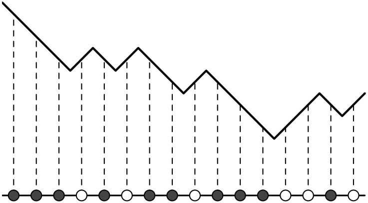

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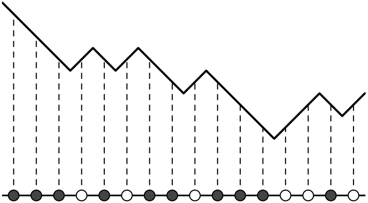

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Initially, i.i.d. Bernoulli($\frac{1}{2}$).
Re-center around the hole-particle pair: TASEP as seen from a second class particle, and its stationary distribution gives ν.

Stationary measure of TASEP as seen from a second class particle:

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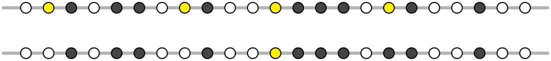

- A stationary measure for TASEP with infinitely many second class particles: a renewal process.
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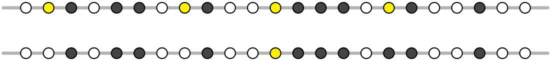

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An alternative description: the corresponding surface is the lower one of two non-intersecting random walks.

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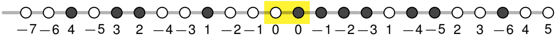
Take the stationary measure (of TASEP as seen from a second class particle):



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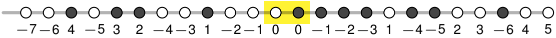

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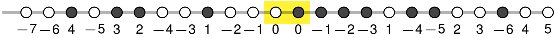

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- Let $\overline{G}((a,b))$ be the time when the particle labeled b is switched with the hole labeled a; let $\overline{\xi}(v) = \overline{G}(v+(1,0)) \wedge \overline{G}(v+(0,1)) - \overline{G}(v)$.

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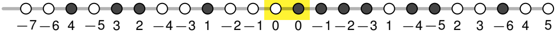

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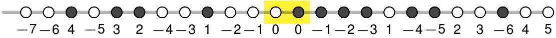

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- ν is given by $\overline{\xi}, \overline{\Gamma}$, reweighted by $\overline{\xi}((0,0))^{-1}$.

Ingredients of the proof

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3. From semi-infinite geodesic to finite geodesics.
4. A uniform convergence in a rectangle:
   Convergence of the law;
   Upgrade in probability convergence to almost surely convergence.

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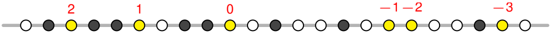

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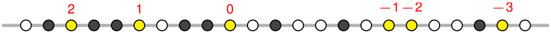

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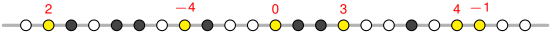

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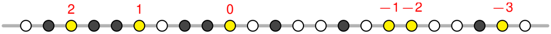

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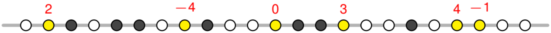

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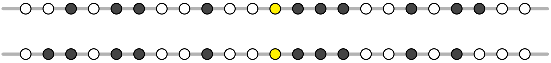

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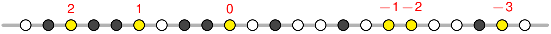

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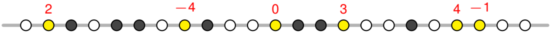

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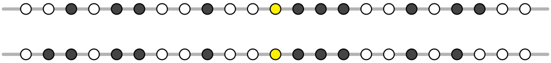
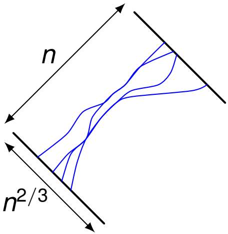


- W.h.p., left are negative and right are positive.

Rectangle uniform convergence



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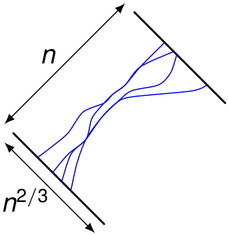

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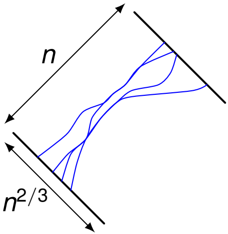

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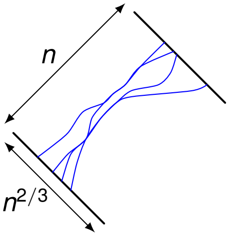

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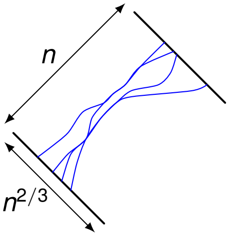

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2. Exponential convergence speed: divide the geodesic into independent segments, each apply the uniform convergence.

Thank you!

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