The Environment Seen from Geodesics in Exponential Last Passage Percolation

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The model: directed LPP with exponential weights
We study the directed last passage percolation (LPP) on $\mathbb{Z}^2$.

- $\xi(v) \sim \text{Exp}(1)$, i.i.d. $\forall v \in \mathbb{Z}^2$
- Passage time: $T_{u,v} := \max_\gamma \sum_{w \in \gamma} \xi(w)$
- Geodesic: $\Gamma_{u,v} := \arg\max_\gamma \sum_{w \in \gamma} \xi(w)$
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- Geodesic: $\Gamma_{u,v} := \arg\max_{\gamma} \sum_{w \in \gamma} \xi(w)$

Equivalent to TASEP, exactly solvable with $1:2:3$ scaling.
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- $2^{-4/3} n^{-1/3} (T_{(0,0),(n,n)} - 4n)$ converges weakly to the GUE Tracy-Widom distribution (Johansson, 2000).

Point to line profile: stationary Airy process minus a parabola (Borodin and Ferrari, 2008)

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General initial data: KPZ fixed point (Matetski, Quastel, and Remenik, 2017).

Joint scaling limit: the directed landscape (Dauvergne, Ortmann, and Virág, 2018).
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$$2^{-4/3} n^{-1/3} \left( T_{(0,0),(n-x(2n)^{2/3},n+x(2n)^{2/3})} - 4n \right) \Rightarrow A_2(x) - x^2$$
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$$2^{-4/3} n^{-1/3} \left( T_{(0,0),(n-x(2n)^{2/3},n+x(2n)^{2/3})} - 4n \right) \Rightarrow \mathcal{A}_2(x) - x^2$$

$\mathcal{A}_2$ is absolute continuous with respect to Brownian motion (Corwin and Hammond, 2014).
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The problem and our results
We study the local behavior along geodesics.
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\[
\xi \{ v \} = \{ \xi (v + u) \} \quad u \in \mathbb{Z}^2 \quad \text{and} \quad \Gamma (0, 0), (n, n) - v \in \{0, 1\} \mathbb{Z}^2.
\]

Consider the environment for all \( v \in \Gamma (0, 0), (n, n) \).

Empirical measure:

\[
\mu_n := \frac{1}{|\Gamma (0, 0), (n, n)|} \sum_{v \in \Gamma (0, 0), (n, n)} \delta (\xi \{ v \}, \Gamma (0, 0), (n, n) - v).
\]
We study the local behavior along geodesics.

Random environment at $v$ contains:

$$\xi\{v\} = \{\xi(v + u)\}_{u \in \mathbb{Z}^2} \in \mathbb{R}^{\mathbb{Z}^2} \text{ and } \Gamma_{(0,0),(n,n)} - v \in \{0, 1\}^{\mathbb{Z}^2}.$$
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Consider the environment for all \( v \in \Gamma_{(0,0),(n,n)} \).
We study the local behavior along geodesics.

- Random environment at $\nu$ contains:
  $$\xi\{\nu\} = \{\xi(\nu + u)\}_{u \in \mathbb{Z}^2} \in \mathbb{R}^{\mathbb{Z}^2}$$ and $\Gamma(0,0),(n,n) - \nu \in \{0,1\}^{\mathbb{Z}^2}$.

- Consider the environment for all $\nu \in \Gamma(0,0),(n,n)$.

- Empirical measure:
  $$\mu_n := \frac{1}{|\Gamma(0,0),(n,n)|} \sum_{\nu \in \Gamma(0,0),(n,n)} \delta(\xi\{\nu\},\Gamma(0,0),(n,n) - \nu)$$.
Main result

- **Empirical measure**

\[ \mu_n := \frac{1}{|\Gamma_{(0,0),(n,n)}|} \sum_{v \in \Gamma_{(0,0),(n,n)}} \delta(\xi\{v\}, \Gamma_{(0,0),(n,n)} - v), \]  

a random measure in the space \( \mathbb{R}^2 \times \{0, 1\}^2 \).
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Question: limiting behavior of \( \mu_n \) as \( n \to \infty \)?
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This was first asked for the first passage percolation (FPP) setting (e.g. Hoffman, AimPL, 2015).
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In a different direction, (Bates, 2020) proved convergence of the empirical measure of weights \( \frac{1}{|\Gamma_{(0,0),(n,n)}|} \sum_{e \in \Gamma_{(0,0),(n,n)}} \delta_{\xi(e)}, \) for certain families of i.i.d. edge weights, using a variational formula.
Empirical measure

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A related question: convergence of the environment at a single point?

- Also consider \( \Gamma_{(0,0)} = \{\Gamma_{(0,0)}[i]\}_{i=1}^{\infty} \), the semi-infinite geodesic in the \((1,1)\) direction; let

\[ \bar{\mu}_r := \frac{1}{r} \sum_{i=1}^{r} \delta(\xi\{\Gamma_{(0,0)}[i]\}, \Gamma_{(0,0)} - \Gamma_{(0,0)}[i]) \cdot \]
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Theorem (Sly and Z., unpublished)

There is a (deterministic) measure \( \nu \), such that

1. \( \mu_n \to \nu \) in probability.
2. The law of \( \xi\{\nu\}, \Gamma(0,0),(n,n) - \nu \) converges to \( \nu \), where \( \nu \) is the midpoint of \( \Gamma(0,0),(n,n) \).
3. \( \overline{\mu}_r \to \nu \) almost surely.
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Next Question: what is the limiting measure \( \nu \)?
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We give an explicit description of the limiting measure $\nu$. 
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Theorem (Martin, Sly, and Z., 2021)

We give an explicit description of the limiting measure $\nu$.

Semi-infinite geodesic $\Leftrightarrow$ Competition interface from stationary $\Leftrightarrow$ TASEP with a second class particle.
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Semi-infinite geodesic $\Leftrightarrow$ Competition interface from stationary $\Leftrightarrow$ TASEP with a second class particle

Similar results hold for geodesics in other directions.
Some applications

This explicit construction enables one to explicitly compute the law \( \nu \), thus all local statistics of the geodesic.

Some examples:

Law of the weight of a vertex on the geodesic: for \( \xi \),

\[ \Gamma_\sim \nu_{\frac{1}{2}n} \left| \{ v \in \Gamma(0,0), (n,n) : \xi(v) > x \} \right| \to P[\xi((0,0)) > x] = (1 + 3x^4 + x^8) e^{-x}. \]

(Note that before we know \( E \xi((0,0)) = 2 \), since \( E T(0,0), (n,n) \sim 4n \).)

Portion of 'turnings' along the geodesic:

Let \( N_n \) be the number of \( v \in \mathbb{Z}^2 \), such that

\[ \{ v, v - (1,0), v + (0,1) \} \subset \Gamma(0,0), (n,n), \text{ or } \{ v, v + (1,0), v - (0,1) \} \subset \Gamma(0,0), (n,n). \]

Then \( N_n \to 3n^2 \) in probability.
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- Law of the weight of a vertex on the geodesic: for $\bar{\xi}, \bar{\Gamma} \sim \nu$

$$\frac{1}{2n} \left| \{ v \in \Gamma_{(0,0),(n,n)} : \xi(v) > x \} \right|$$

$$\rightarrow \mathbb{P}[\bar{\xi}((0,0)) > x] = \left( 1 + \frac{3x}{4} + \frac{x^2}{8} \right) e^{-x}.$$ 

(Note that before we know $\mathbb{E} \bar{\xi}((0,0)) = 2$, since $\mathbb{E} T_{(0,0),(n,n)} \sim 4n$.)
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- Portion of ‘turnings’ along the geodesic:

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  \]

  Then $\frac{N_n}{2n} \rightarrow \frac{3}{8}$ in probability.
The limiting measure

Semi-infinite geodesic ⇔ Competition interface from stationary
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(0, 0)
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LPP geodesic environment
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Semi-infinite geodesic ⇐ Competition interface from stationary

Γ_{(0,0)}

\textbf{Z}

(0, 0)

⇒ \xi(v) = G(v + (1, 0)) \land G(v + (0, 1)) - G(v).

Boundary of \textbf{I} = \{v: G(v) \leq 0\} is a (two-sided) simple random walk. Define \xi_v(\lor (v)) = G(v) - G(v - (1, 0)) \lor G(v - (0, 1)).

Given \textbf{I}, \{\xi_v(\lor (v))\}_{v \notin \textbf{I}} are i.i.d. \text{Exp}(1).

Let the aggregate at time \textbf{t} be \{v: G(v) \leq t\}. Then \xi_v(\lor (v)) is the waiting time at v, and \textbf{Z} = Γ_{(0,0)} + (1/2, 1/2).
Semi-infinite geodesic $\iff$ Competition interface from stationary $(0,0)$

Busemann function $G(v) = \lim_{n \to \infty} T_{(0,0),(n,n)} - T_{v,(n,n)}$. 
The limiting measure

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Boundary of $I = \{v : G(v) \leq 0\}$ is a (two-sided) simple random walk. Define $\xi^\wedge(v) = G(v) - G(v - (1,0)) \lor G(v - (0,1))$. Given $I$, $\{\xi^\wedge(v)\}_{v \notin I}$ are i.i.d. $\text{Exp}(1)$. (Seppäläinen, etc.)
The limiting measure

Semi-infinite geodesic ⇔ Competition interface from stationary

\[ Busemann \text{ function } G(v) = \lim_{n \to \infty} T_{(0,0),(n,n)} - T_{v,(n,n)}. \]
\[ \Rightarrow \xi(v) = G(v + (1,0)) \land G(v + (0,1)) - G(v). \]
\[ \text{Boundary of } I = \{ v : G(v) \leq 0 \} \text{ is a (two-sided) simple random walk. Define } \xi^\vee(v) = G(v) - G(v - (1,0)) \lor G(v - (0,1)). \]
\[ \text{Given } I, \{ \xi^\vee(v) \}_{v \notin I} \text{ are i.i.d. Exp}(1). \text{ (Seppäläinen, etc.)} \]
\[ \text{Let the aggregate at time } t \text{ be } \{ v : G(v) \leq t \}. \text{ Then } \xi^\vee(v) \text{ is the waiting time at } v, \text{ and } Z = \Gamma_{(0,0)} + \left( \frac{1}{2}, \frac{1}{2} \right). \]
The limiting measure

Semi-infinite geodesic ⇔ Competition interface from stationary

The other direction: first take the two species corner growth process, where the initial boundary is given by a (two-sided) simple random walk.

- Let $G(v)$ be the time when $v$ is occupied.
- Let $\Gamma(0,0) = Z - (\frac{1}{2}, \frac{1}{2})$, and
  
  $\xi(v) = G(v + (1,0)) \wedge G(v + (0,1)) - G(v)$.
The limiting measure

Semi-infinite geodesic \Leftrightarrow \text{Competition interface from stationary}

\( \Gamma_{(0,0)} \)

The other direction: first take the two species corner growth process, where the initial boundary is given by a (two-sided) simple random walk.

- Let \( G(\nu) \) be the time when \( \nu \) is occupied.
- Let \( \Gamma_{(0,0)} = Z - \left( \frac{1}{2}, \frac{1}{2} \right) \), and
  \[ \xi(\nu) = G(\nu + (1,0)) \wedge G(\nu + (0,1)) - G(\nu). \]
- Now suffices to study local environment around the competition interface.
The limiting measure

Competition interface $\Leftrightarrow$ TASEP with a second class particle
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Competition interface $\Leftrightarrow$ TASEP with a second class particle

- TASEP and growing surface:
The limiting measure

Competition interface $\Leftrightarrow$ TASEP with a second class particle

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![Graphical representation of TASEP with a second class particle and growing surface]
The limiting measure

Competition interface $\Leftrightarrow$ TASEP with a second class particle

- TASEP and growing surface:

Now keep track of a hole-particle pair:
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Competition interface ⇔ TASEP with a second class particle

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Initially, i.i.d. Bernoulli($\frac{1}{2}$).
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Competition interface ⇔ TASEP with a second class particle

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Initially, i.i.d. Bernoulli(\(\frac{1}{2}\)).
Re-center around the hole-particle pair: TASEP as seen from a second class particle, and its stationary distribution gives \(\nu\).
Stationary measure of TASEP as seen from a second class particle:
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\[ \begin{array}{ccccccccccc}
* & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * \\
\end{array} \]

- A stationary measure for TASEP with infinitely many second class particles: a renewal process.
- Identify 2CP to the right with particles, and 2CP to the left with holes. (Ferrari, Fontes, and Kohayakawa, 1994)
Stationary measure of TASEP as seen from a second class particle:

A stationary measure for TASEP with infinitely many second class particles: a renewal process.

Identify 2CP to the right with particles, and 2CP to the left with holes. (Ferrari, Fontes, and Kohayakawa, 1994)

An alternative description: the corresponding surface is the lower one of two non-intersecting random walks.
The limiting measure

Now we construct $\nu$. 
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Take the stationary measure (of TASEP as seen from a second class particle):

$$
\begin{array}{cccccccccccc}
-7 & 6 & 4 & -5 & 3 & 2 & -4 & 3 & 1 & -2 & -1 & 0 & 0 & -1 & 2 & -3 & 1 & -4 & 5 & 2 & 3 & -6 & 4 & 5
\end{array}
$$
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\end{array}$

2CP $\Rightarrow$ a hole-particle pair, label all particles and holes.
Now we construct $\nu$.

Take the stationary measure (of TASEP as seen from a second class particle):

$-7-6-5-4-3-2-1-0-1-2-3-1-4-5-2-3-6-4-5$

- 2CP $\Rightarrow$ a hole-particle pair, label all particles and holes.
- Let $\overline{G}((a, b))$ be the time when the particle labeled $b$ is switched with the hole labeled $a$; let $\overline{\xi}(v) = \overline{G}(v + (1, 0)) \wedge \overline{G}(v + (0, 1)) - \overline{G}(v)$. 
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- Let $\overline{G}((a, b))$ be the time when the particle labeled $b$ is switched with the hole labeled $a$; let $\overline{\xi}(v) = \overline{G}(v + (1, 0)) \wedge \overline{G}(v + (0, 1)) - \overline{G}(v)$.
- Let $\Gamma$ consist of all $(a, b)$, which are the labels for the hole-particle pair at some time.
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2CP $\Rightarrow$ a hole-particle pair, label all particles and holes.

Let $\overline{G}((a, b))$ be the time when the particle labeled $b$ is switched with the hole labeled $a$; let

$$\bar{\xi}(\nu) = \overline{G}(\nu + (1, 0)) \land \overline{G}(\nu + (0, 1)) - \overline{G}(\nu).$$

Let $\overline{\Gamma}$ consist of all $(a, b)$, which are the labels for the hole-particle pair at some time.

$\nu$ is given by $\bar{\xi}, \overline{\Gamma}$, reweighted by $\bar{\xi}((0, 0))^{-1}$. 
Ingredients of the proof
General structure of arguments

Main steps:

1. Convergence of TASEP as seen from a second class particle: Initial i.i.d. Bernoulli corresponds to a semi-infinite geodesic. Converge to the stationary measure.
2. Convergence of empirical distribution: Ergodicity of the stationary process.
3. From semi-infinite geodesic to finite geodesics.
4. A uniform convergence in a rectangle: Convergence of the law; Upgrade in probability convergence to almost surely convergence.
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Consider two TASEP as seen from a second class particle:

1. $\Psi$: the stationary one;
2. $\Phi_t$: start with i.i.d. Bernoulli($\frac{1}{2}$), and run for time $t$. 

Observation: consider TASEP with infinitely many 2CP, under stationary:
- Left to holes, right to particles $\Rightarrow$ the stationary measure $\Psi$.
- Left to particles, right to holes $\Rightarrow$ i.i.d. Bernoulli($\frac{1}{2}$), i.e. $\Phi_0$.

Initially, label all 2CP with $Z$, from right to left.
- Rule: larger labels are stronger. Run for time $t$.
  - Left to holes, right to particles $\Rightarrow$ the stationary measure $\Psi$.
  - Negative to holes, positive to particles $\Rightarrow$ $\Phi_t$.

W.h.p., left are negative and right are positive.
Consider two TASEP as seen from a second class particle:

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Show that (an average of) $\Phi_t$ is similar to $\Psi$: a coupling.
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Convergence of TASEP

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![Diagram of TASEP](image)
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Usage: uniform convergence implies that the empirical distribution in part of the geodesic is also close to $\nu$. 

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  Also need: the laws for vertices (in the geodesic) at distances $o(n)$ are close.
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1. Convergence of law:
   Also need: the laws for vertices (in the geodesic) at distances $o(n)$ are close.

2. Exponential convergence speed: divide the geodesic into independent segments, each apply the uniform convergence.
Thank you!


