Interlacing adjacent levels of β-Jacobi corners processes

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Plan of today

Problem setup
- β-Jacobi corners processes
- Limit scheme and some prior results

Main results: LLM and CLT of interlacing adjacent levels

Interpretations and implications
- LLN: digram and roots of Jacobi polynomials
- CLT: pullback of GFF

Key techniques used
- Macdonald processes
- Differential operators
- Dimension reduction
- A Gaussian type asymptote
- To actual Gaussianity

Model: MANOVA matrices & Jacobi ensemble

Let X be $A \times M$, $A \geq M$, Y be $N \times M$ random matrices, every entry i.i.d real, complex, or quaternion Gaussian. The distribution of $X^*X(X^*X + Y^*Y)^{-1}$ is the *MANOVA ensemble*.

(Almost surely) it has $K = \min\{M, N\}$ eigenvalues different from 0 and 1. The distribution is the K-particle Jacobi ensemble:

$$\prod_{1 \leq i < j \leq K} (x_i - x_j)^\beta \prod_{i=1}^{K} x_i^p (1 - x_i)^q$$

for $p = \frac{\beta}{2}(A - M + 1) - 1$, $q = \frac{\beta}{2}(|M - N| + 1) - 1$, and $\beta = 1, 2, 4$, corresponding to real, complex, or quaternion entries.

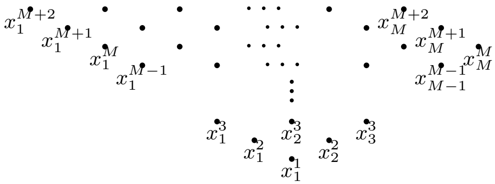
Consider a multilevel setting

Let χ^M be the set of infinite families of sequences $x^1, x^2, \cdots$, where for each $N \geq 1$, x^N is an increasing sequence with length $\min(N, M)$:

$$0 \leq x_1^N < \cdots < x_{\min(N,M)}^N \leq 1$$

and for each $N > 1$, x^N and x^{N-1} interlace:

$$x_1^N < x_1^{N-1} < x_2^N < \cdots .$$



Corners processes

The β–Jacobi corners process, first introduced in [BG15], is a random element of χ^M with distribution $\mathbb{P}^{\alpha,M,\theta}$, given in the following way: the marginal distribution of a single x^N has density (with respect to Lebesgue measure) proportional to

$$\prod_{1\le i<j\le \min(N,M)} (x_i^N - x_j^N)^{2\theta} \prod_{i=1}^{\min(N,M)} (x_i^N)^{\theta\alpha-1}(1-x_i^N)^{\theta(|M-N|+1)-1},$$

and a specified conditional distribution of x^{N-1} given x^N (see [BG15, Section 2.3] for a complete definition).

Matrix model for multilevel ensemble

For $\beta = 1, 2, 4$ there are many ways to obtain the β-Jacobi ensemble, and many can be extended to the multilevel setting.

Consider infinite random matrices X and Y, let X^{AM} be the $A \times M$ top–left corner of X, and Y^{NM} the $N \times M$ top–left corner of Y. Denote

$$\mathcal{M}^{ANM} = (X^{AM})^* X^{AM} \left((X^{AM})^* X^{AM} + (Y^{NM})^* Y^{NM} \right)^{-1},$$

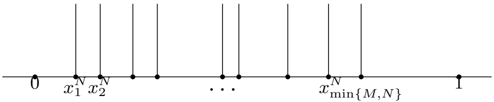
It was proved in [Sun16] that the joint distribution of (different from $0, 1$) eigenvalues in $\mathcal{M}^{AnM}$, $n = 1, \cdots, N$, for real and complex entries, is the same as the first N rows of β-Jacobi corners process with $\alpha = A - M + 1$, and $\theta = \frac{\beta}{2}$, for $\beta = 1, 2$ respectively.

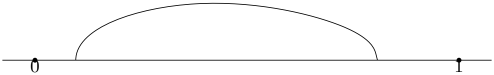
Passing $\alpha, M, N \to \infty$

Consider level N in $\mathbb{P}^{\alpha,M,\theta}$. Let the parameters α and M and level N depend on a large auxiliary variable $L \to \infty$:

$$\lim_{L\to\infty} \frac{\alpha}{L} = \hat{\alpha}, \lim_{L\to\infty} \frac{N}{L} = \hat{N}, \lim_{L\to\infty} \frac{M}{L} = \hat{M}.$$

Then the random sequence $x_1^N \le \cdots \le x_{\min\{M,N\}}^N$, or the measure $L^{-1}\sum_{i=1}^{\min\{M,N\}} \delta_{x_i^N}$, converges to a random function





Known asymptote results

Law of Large Numbers (classical result) for (smooth) function f, there is

$$\lim_{L\to\infty} L^{-1} \sum_{i=1}^{\min\{M,N\}} f(x_i^N) = \int_0^1 \phi(x) f(x) dx$$

in probability. Here $\phi : [0,1] \to \mathbb{R}$ is an explicit deterministic function. (see, e.g. [Kil08], [DP12], [BG15]). This is an analogue of Wigner semicircle law [Wig58] (which is in Hermite ensemble).

Central Limit Theorem the sum

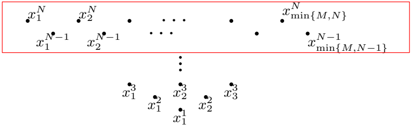
$$\sum_{i=1}^{\min\{M,N\}} f(x_i^N) - \mathbb{E}\left(f(x_i^N)\right)$$

converges to Gaussian as $L \to \infty$.

- $\beta = 1, 2, 4$: classical. see e.g. [Sze52] [For10].
- General β, first by Johansson for Hermitian matrices [Joh98].
- General β in the Jacobi case, recently by Dumitriu and Paquette [DP12].
- Multilevel setting, the joint convergence to Gaussian was proved by Borodin and Gorin [BG15].

Our problem: adjacent levels

A sequential construction: $\sum_{n=1}^{N}\left(\sum_{i=1}^{\min\{M,n\}} f(x_i^n) - \sum_{i=1}^{\min\{M,n-1\}} f(x_i^{n-1})\right)$.



When $N > M$, denote $x_i^N = 1$ for any $N < i \leq M$.
Denote $\mathfrak{P}_k(x^N) = \sum_{i=1}^{N}(x_i^N)^k$ to be the moments.

Theorem (LLN of moments)

The random variable $\mathfrak{P}_k(x^N) - \mathfrak{P}_k(x^{N-1})$ converges to a constant as $L \to \infty$, in the sense that the variance decays in $O(L^{-1})$. The constant is given by the following contour integral:

$$\lim_{L\to\infty} \mathbb{E}\left(\mathfrak{P}_k(x^N) - \mathfrak{P}_k(x^{N-1})\right) = \frac{1}{2\pi \mathbf{i}} \oint \left(\frac{v}{v+\hat{N}} \cdot \frac{v-\hat{\alpha}}{v-\hat{\alpha}-\hat{M}}\right)^k \frac{1}{v+\hat{N}} dv,$$

where the integration contour encloses the pole at $-\hat{N}$ but not $\hat{\alpha}+\hat{M}$, and is positively oriented.

Fluctuation: discrete Gaussianity

There are two CLT of fluctuation, when considering discrete levels or an integral cross different levels, and they are in different scales: for discrete level it is $L^{\frac{1}{2}}$, for an integral it is L.

Theorem (CLT of discrete levels)

The random vector

$$L^{\frac{1}{2}} \left(\mathfrak{P}_{k_i}(x^{N_i}) - \mathfrak{P}_{k_i}(x^{N_i - 1}) - \mathbb{E} \left(\mathfrak{P}_{k_i}(x^{N_i}) - \mathfrak{P}_{k_i}(x^{N_i - 1}) \right) \right)_{i=1}^{h}$$

converges to centered a Gaussian random vector, whose covariance between the ith and jth component is

$$-\delta_{\hat{N}_i = \hat{N}_j} \cdot \frac{k_i k_j}{k_i + k_j} \cdot \frac{\theta^{-1}}{2\pi \mathbf{i}} \oint \frac{1}{(v + \hat{N}_i)^2} \left(\frac{v}{v + \hat{N}_i} \cdot \frac{v - \hat{\alpha}}{v - \hat{\alpha} - \hat{M}} \right)^{k_i + k_j} dv,$$

where the contour encloses $-\hat{N}_i$ but not $\hat{\alpha} + \hat{M}$.

Fluctuation: discrete Gaussianity (cont.)

In [BG15], it was shown that the random vector

$$\left(\mathfrak{P}_{k_i'}(x^{N_i'}) - \mathbb{E}\left(\mathfrak{P}_{k_i'}(x^{N_i'})\right)\right)_{i=1}^{h'}$$

converge (as $L \to \infty$) to centered Gaussian whose covariance between the ith and jth component is

$$\frac{\theta^{-1}}{(2\pi\mathbf{i})^2} \oint \oint \frac{1}{(v_1 - v_2)^2} \prod_{i=1}^{2} \left(\frac{v_i}{v_i + \hat{N}_i} \cdot \frac{v_i - \hat{\alpha}}{v_i - \hat{\alpha} - \hat{M}} \right)^{k_i} dv_i.$$

Here we show that the convergence of both random vectors are joint, but they are asymptptotically independent.

Fluctuation: smooth Gaussianity

Theorem (CLT of integral over levels)

Let $g_1, \cdots, g_h \in L^\infty([0,1])$ continuous almost everywhere. As $L \to \infty$, the random vector

$$\left(L \int_0^1 g_i(y) \left(\mathfrak{P}_{k_i}(x^{\lfloor Ly \rfloor}) - \mathfrak{P}_{k_i}(x^{\lfloor Ly \rfloor - 1}) - \mathbb{E}\left(\mathfrak{P}_{k_i}(x^{\lfloor Ly \rfloor}) - \mathfrak{P}_{k_i}(x^{\lfloor Ly \rfloor - 1}) \right) \right) dy \right)_{i=1}^{h}$$

converges jointly in distribution to a centered Gaussian vector, with covariance between the ith and jth component is given by

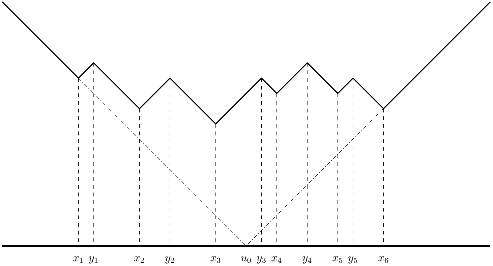
Fluctuation: smooth Gaussianity (cont.)

$$\iint_{0\le y_1<y_2\le 1} \frac{\theta^{-1}}{(2\pi \mathbf{i})^2} \oint\oint \frac{k_i k_j}{(v_1-v_2)^2(v_1+y_1)(v_2+y_2)}$$
$$\times \Bigg(g_i(y_1)g_j(y_2)\left(\frac{v_1}{v_1+y_1}\cdot\frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}}\right)^{k_i}\left(\frac{v_2}{v_2+y_2}\cdot\frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}}\right)^{k_j}$$
$$+g_j(y_1)g_i(y_2)\left(\frac{v_1}{v_1+y_1}\cdot\frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}}\right)^{k_j}\left(\frac{v_2}{v_2+y_2}\cdot\frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}}\right)^{k_i}\Bigg) dv_1dv_2dy_1dy_2$$
$$-\int_0^1 \frac{\theta^{-1}}{2\pi\mathbf{i}}\oint \frac{g_i(y)g_j(y)k_ik_j}{(k_i+k_j)(v+y)^2}\left(\frac{v}{v+y}\cdot\frac{v-\hat{\alpha}}{v-\hat{\alpha}-\hat{M}}\right)^{k_i+k_j} dvdy,$$

where in the first integral, the contours are nested: $|v_1| \ll |v_2|$, and enclose $-y_1, -y_2$ but not $\hat{\alpha}+\hat{M}$; in the second integral, the contour encloses $-y$ but not $\hat{\alpha}+\hat{M}$.

Kerov's diagram

Each interlacing sequence corresponds to a diagram:



Theorem (Convergence of diagram)

Let $w^{x^N, x^{N-1}}$ be the interlacing diagram of the sequence x^N, x^{N-1}. Then it converges to a deterministic diagram φ in the sense that, in probability,

$$\lim_{L\to\infty} \sup_{u\in\mathbb{R}} \left| w^{x^N, x^{N-1}}(u) - \varphi(u) \right| = 0.$$

Convergence of measure

Consider the signed measure $\sum_{i=1}^{N} \delta_{x_i^N} - \sum_{i=1}^{N-1} \delta_{x_i^{N-1}}$, as $L \to \infty$.

Theorem (LLN of the measure)

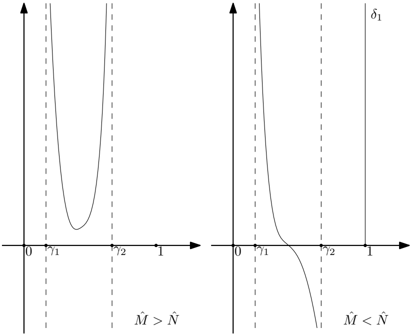
For any differentiable $f : [0,1] \to \mathbb{R}$, the random variable

$$\sum_{i=1}^{N} f(x_i^N) - \sum_{i=1}^{N-1} f(x_i^{N-1})$$

converges (in probability) to constant $\int_0^1 f(u)\tau(u)du$, as $L \to \infty$. Here $\tau : \mathbb{R} \to \mathbb{R}$ is defined as

$$\tau(u) = \begin{cases} \frac{\hat{M}-\hat{N}+(\hat{N}+\hat{M}+\hat{\alpha})(1-u)}{2\pi(\hat{N}+\hat{M}+\hat{\alpha})(1-u)} \frac{1}{\sqrt{(\gamma_2-u)(u-\gamma_1)}}, & u \in (\gamma_1, \gamma_2) \\ C(\hat{M}, \hat{N})\delta(u-1), & u \in (-\infty, \gamma_1] \bigcup [\gamma_2, \infty). \end{cases}$$

Convergence of measure (cont.)



$$\gamma_{1,2} = \frac{\left(\sqrt{(\hat{\alpha}+\hat{M})(\hat{\alpha}+\hat{N})} \mp \sqrt{\hat{M}\hat{N}}\right)^2}{(\hat{N}+\hat{M}+\hat{\alpha})^2},$$

$$C(\hat{M},\hat{N}) = \begin{cases} 0, & \hat{M} > \hat{N} \\ \frac{1}{2}, & \hat{M} = \hat{N} \\ 1, & \hat{M} < \hat{N} \end{cases}.$$

- Total measure $\int \tau = 1$.
- $\hat{M} < \hat{N}$, delta function at 1.
- $\tau = 0$ outside $(\gamma_1, \gamma_2) \bigcup \{1\}$.
- $\varphi'' = 2\tau$.
- Not true for non-smooth f: e.g. an indicator function of an interval.

Sending $\theta \to 0$

There is a limit transition between β-Jacobi corners processes and the roots of the Jacobi orthogonal polynomials.

Let $\mathcal{F}_n^{p,q}$ be the Jacobi orthogonal polynomials of degree n with weight function $x^p(1-x)^q$ on $[0,1]$. Let $j_{M,N,\alpha,i}$ be the ith root (in increasing order) of $\mathcal{F}_{\min(M,N)}^{\alpha-1,|M-N|}$, for $1 \le i \le \min(M,N)$. We also denote $j_{M,N,\alpha,i} = 1$, for any fixed M, N, α, and $\min(M,N) < i \le N$.

Theorem ([BG15, Theorem 5.1])

Let $(x^1, x^2, \cdots) \in \chi^M$ be distributed as $\mathbb{P}^{\alpha,M,\theta}$, and let $j_{M,N,\alpha,i}$ be the ith root (in increasing order) of $\mathcal{F}_{\min(M,N)}^{\alpha-1,|M-N|}$, for $1 \le i \le \min(M,N)$. Then there is

$$\lim_{\theta\to\infty} x_i^N = j_{M,N,\alpha,i},$$

in probability.

Roots of Jacobi polynomials

With the transition, and our LLN above, it is easy to obtain that

Theorem (Convergence of roots)

There is an interlacing relationship for the roots:

$$j_{M,N,\alpha,1} \leq j_{M,N-1,\alpha,1} \leq j_{M,N,\alpha,2} \leq \cdots .$$

Then diagram corresponding to this interlacing sequence uniformly converges to φ, as $L \to \infty$.

Recall the definition of GFF

The *Gaussian Free Field* with Dirichlet boundary conditions in the upper half plane $\mathbb{H}$ is defined as a mean 0 (generalized) Gaussian random field $\mathcal{G}$ on $\mathbb{H}$, whose covariance (for any $z, w \in \mathbb{H}$) is

$$\mathbb{E}(\mathcal{G}(z)\mathcal{G}(w)) = -\frac{1}{2\pi} \ln \left| \frac{z-w}{z-\bar{w}} \right|.$$

Since it has a singulrity at the diagonal $z = w$, the value of the GFF at a point is not defined, however, it can be well-defined as an element of a certain functional space. In particular, the integrals of $\mathcal{G}(z)$ against sufficiently smooth measures are genuine Gaussian random variables.

The pullback

We connect the upper half plane with the area where the β-Jacobi ensemble lives. This was introduced in [BG15].
Let $D \subset [0,1] \times \mathbb{R}_{>0}$ be defined by the following inequality

$$\left| x - \frac{\hat{M}\hat{N} + (\hat{M}+\hat{\alpha})(\hat{N}+\hat{\alpha})}{(\hat{N}+\hat{\alpha}+\hat{M})^2} \right| \leq \frac{2\sqrt{\hat{M}\hat{N}(\hat{M}+\hat{\alpha})(\hat{N}+\hat{\alpha})}}{(\hat{N}+\hat{\alpha}+\hat{M})^2}.$$

Let $\Omega : D \bigcup \{\infty\} \to \mathbb{H} \bigcup \{\infty\}$ such that the horizontal section of D at height $\hat{N}$ is mapped to the half-plane part of the circle, centered at

$$\frac{\hat{N}(\hat{\alpha}+\hat{M})}{\hat{N}-\hat{M}}$$

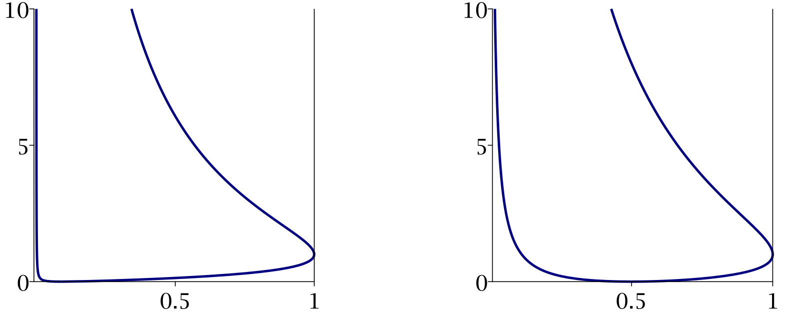
with radius

$$\frac{\sqrt{\hat{M}\hat{N}(\hat{M}+\hat{\alpha})(\hat{N}+\hat{\alpha})}}{\left|\hat{N}-\hat{M}\right|}$$

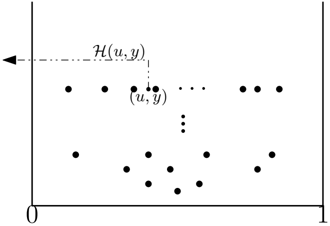
The pullback (cont.)

(when $\hat{N} = \hat{M}$ the circle is replaced by the vertical line at $\frac{\hat{\alpha}}{2}$), and point $u \in \mathbb{H}$ is the image of

$$\left(\frac{u}{u+\hat{N}} \cdot \frac{u-\hat{\alpha}}{u-\hat{\alpha}-\hat{M}}, \hat{N} \right).$$



Identification of the limit object



For any $(u, y) \in [0, 1] \times \mathbb{R}_{>0}$, define $\mathcal{H}(u, y)$ to be the number of i such that $x_i^{\lfloor y \rfloor}$ is less than u.

Let $\mathcal{K}$ be the generalized Gaussian random field in $[0, 1] \times \mathbb{R}_{\geq 0}$ which is 0 outside D and is equal to $\mathcal{G} \circ \Omega$ (i.e. the pullback of $\mathcal{G}$ with respect to map Ω) inside D.

In [BG15], it was proved that the function $\mathcal{H}(u, Ly)$ converges to the random field $\mathcal{K}$.

Identification of the limit object: discrete levels

For $y > 1$, let $\mathcal{W}(u, y) = \mathcal{H}(u, y) - \mathcal{H}(u, y-1)$. Then it is expected that the function $\mathcal{W}$ converges to some derivative of the random field $\mathcal{K}$.

Theorem (Discrete levels, "half" derivative)

As $L \to \infty$, for any integers $k_1, \cdots, k_h$, and real numbers $0 < \hat{N}_1 \leq \cdots \leq \hat{N}_h$, the distribution of the vector

$$\left(L^{\frac{1}{2}} \int_0^1 u^{k_i} \left(\mathcal{W}(u, L\hat{N}_i) - \mathbb{E}\left(\mathcal{W}(u, L\hat{N}_i) \right) \right) du \right)_{i=1}^{h}$$

converges weakly to a joint Gaussian distribution, which is the same as the weak limit

$$\lim_{\delta \to 0+} \delta^{-\frac{1}{2}} \left(\int_0^1 u^{k_i} \mathcal{K}(u, \hat{N}_i + \delta) du - \int_0^1 u^{k_i} \mathcal{K}(u, \hat{N}_i) du \right)_{i=1}^{h}.$$

Identification of the limit object: discrete levels (cont.)

For any integers $k_1, \cdots, k_h$ *and* $k'_1, \cdots, k'_{h'}$*, real numbers* $0 < \hat{N}_1 \leq \cdots \leq \hat{N}_h$ *and* $0 < \hat{N}'_1 \leq \cdots \leq \hat{N}'_{h'}$*, the convergence of the above vector and*

$$\left(\int_0^1 u^{k'_i} \left(\mathcal{H}(u, L\hat{N}'_i) - \mathbb{E}\left(\mathcal{H}(u, L\hat{N}'_i) \right) \right) du \right)_{i=1}^{h'}$$

is jointly, while the limit vectors are independent.

This is different in the case of integral cross levels.

Identification of the limit object: integral cross levels

For any $g \in C^\infty([0,1])$, with $g(1) = 0$, define

$$\mathfrak{Z}_{g,k} = \int_0^1 \int_0^1 u^k g'(y) \mathcal{K}(u,y) dudy.$$

We can extend this definition to $g \in L^2([0,1])$, by a convergence argument.

Theorem (Integral cross levels)

Let $k_1, \cdots, k_h$ be positive integers and $g_1, \cdots, g_h \in L^\infty([0,1])$, each continuous almost everywhere. As $L \to \infty$, the distribution of the vector

$$\left(L \int_0^1 \int_0^1 u^{k_i} g_i(y) \left(\mathcal{W}(u, Ly) - \mathbb{E}\left(\mathcal{W}(u, Ly)\right)\right) dudy \right)_{i=1}^h$$

converges weakly to the distribution of the vector $(\mathfrak{Z}_{g_i,k_i})_{i=1}^h$.

Identification of the limit object: integral cross levels (cont.)

Moreover, take differentiable functions $\tilde{g}_1, \cdots, \tilde{g}_{h'} \in L^\infty([0,1])$, *such that* $\tilde{g}_i(1) = 0$ *and* $\tilde{g}_i' \in L^\infty([0,1])$ *for each* $1 \le i \le h'$, *and positive integers* $k_1', \cdots, k_{h'}'$. *Then the distribution of the vector*

$$\left(\int_0^1 \int_0^1 -u^{k_i'} \tilde{g}_i'(y) \left(\mathcal{H}(u, Ly) - \mathbb{E}\left(\mathcal{H}(u, Ly)\right)\right) dudy \right)_{i=1}^{h'}$$

converges weakly to the distribution of the vector $\left(\mathfrak{Z}_{\tilde{g}_i, k_i'}\right)_{i=1}^{h'}$, *as* $L \to \infty$; *and the convergence of both vectors are joint.*

Remark

There is no a priory reason why such an upgrade for the CLT should hold. e.g. Erdos and Schroder[ES16] show that this is not the case for general Wigner matrices; and in that article the limit might even fail to be Gaussian.

Macdonald processes: the definition

Let $\mathbb{Y}$ be the set of partitions/Young diagrams/infinite non-increasing sequence of non-negative integers, which are eventually zero. And let $\mathbb{Y}_N \subset \mathbb{Y}$ consists of sequences λ such that $\lambda_{N+1} = 0$

Let Ψ^M be the set of all infinite families of sequences $\{\lambda^i\}_{i=1}^{\infty}$, which satisfy

1. For $N \geq 1$, $\lambda^N \in \mathbb{Y}_{\min\{M,N\}}$.
2. For $N \geq 2$, the sequences λ^N and λ^{N-1} interlace: $\lambda_1^N \geq \lambda_1^{N-1} \geq \lambda_2^N \geq \cdots$.

Macdonald processes: the definition (cont.)

The infinite ascending *Macdonald process* with positive parameters $M \in \mathbb{Z}$, $\{a_i\}_{i=1}^{\infty}$, $\{b_i\}_{i=1}^{M}$, $0 < a_i < 1$, $0 < b_i < 1$, is the distribution on Ψ^M, such that the marginal distribution for λ^N is

$$\mathrm{Prob}(\lambda^N = \mu) = \prod_{1 \leq i \leq N, 1 \leq j \leq M} \frac{\prod_{k=1}^{\infty}(1 - a_i b_j q^{k-1})}{\prod_{k=1}^{\infty}(1 - t a_i b_j q^{k-1})} P_\mu(a_1, \cdots, a_N; q, t) Q_\mu(b_1, \cdots, b_M; q, t),$$

and $\{\lambda^N\}_{N\geq 1}$ is a trajectory of a Markov chain with (backward) transition probabilities

$$\mathrm{Prob}(\lambda^{N-1} = \mu | \lambda^N = \nu) = P_{\nu/\mu}(a_N; q, t) \frac{P_\mu(a_1, \cdots, a_{N-1}; q, t)}{P_\nu(a_1, \cdots, a_N; q, t)}.$$

The limit transition to β-Jacobi corner processes

Theorem ([BG15, Theorem 2.8])

Given positive parameters $M \in \mathbb{Z}$, and α, θ. Let random family of sequences $\{\lambda^i\}_{i=1}^{\infty}$, which takes value in Ψ^M, be distributed according to Macdonald process with parameters M, $\{a_i\}_{i=1}^{\infty}$, $\{b_i\}_{i=1}^{M}$. For $\epsilon > 0$, set

$$\begin{aligned} a_i &= t^{i-1}, \quad i = 1, 2, \cdots, \\ b_i &= t^{\alpha+i-1}, \quad i = 1, 2, \cdots, \\ q &= \exp(-\epsilon), \quad t = \exp(-\theta\epsilon) \\ x_j^i(\epsilon) &= \exp(-\epsilon\lambda_j^i) \quad i = 1, 2, \cdots, 1 \le j \le \min\{m, n\}, \end{aligned}$$

then as $\epsilon \to 0$, the distribution of $x^1, x^2, \cdots$ weakly converges to $\mathbb{P}^{\alpha, M, \theta}$.

Similar to [BG15], the idea to compute the moments for Macdonald process, then pass to β-Jacobi corner processes.

An algebraic result from Shuffle algebra

We use another class of operators, which acts on symmetric functions to extract moments (from Macdonald process). These operators were first used in [FD16, Appendix A].

Define $\tilde{\Lambda}$ to be the ring of symmetric formal power series with complex coefficients in countably many variables $x_1, x_2, \cdots$ Let $\mathbf{D}_{-n} : \tilde{\Lambda} \to \tilde{\Lambda}$, such that

$$\mathbf{D}_{-n}\left(\sum_{\lambda\in\mathbb{Y}} c_\lambda P_\lambda(\cdot; q, t)\right) := \sum_{\lambda\in\mathbb{Y}} c_\lambda \left((1-t^{-n})\sum_{i=1}^{\infty}(q^{\lambda_i}t^{-i+1})^n\right) P_\lambda(\cdot; q, t).$$

Namely, the Macdonald polynomials are the eigenvectors for this class of operators.

An algebraic result from Shuffle algebra (cont.)

There is an integral formula for the eigen-operators, see e.g. [Neg13, Theorem 1.2]:

$$\mathbf{D}_{-n} = \frac{(-1)^{n-1}}{(2\pi\mathbf{i})^n} \oint \cdots \oint \frac{\sum_{i=1}^{n} \frac{z_n t^{n-i}}{z_i q^{n-i}}}{\left(1 - \frac{tz_2}{qz_1}\right) \cdots \left(1 - \frac{tz_n}{qz_{n-1}}\right)} \prod_{i<j} \frac{\left(1 - \frac{z_i}{z_j}\right)\left(1 - \frac{qz_i}{tz_j}\right)}{\left(1 - \frac{z_i}{tz_j}\right)\left(1 - \frac{qz_i}{z_j}\right)}$$
$$\times \exp\left(\sum_{k=1}^{\infty} q^k (1 - t^{-k}) \frac{z_1^{-k} + \cdots + z_n^{-k}}{k} p_k\right) \exp\left(\sum_{k=1}^{\infty} (z_1^k + \cdots z_n^k)(1 - q^{-k}) \frac{\partial}{\partial p_k}\right)$$
$$\times \prod_{i=1}^{n} z_i^{-1} dz_i,$$

where p_k is operator of multiplying $p_k \in \tilde{\Lambda}$, and $\frac{\partial}{\partial p_k}$ is its adjoint operator. The contours are understood as taking the coefficient of $(z_1 \cdots z_n)^{-1}$, large and nested as $|z_i| < |tz_{i+1}|$ for each $1 \leq i \leq n-1$.

Apply to special functions

Let $f : B_r \to \mathbb{C}$ be analytic, such that $f(0) \neq 0$; and $g : B_{r'} \to \mathbb{C}$ such that $g(z)f(q^{-1}z) = f(z)$ for any $z \in B_r$.

$$\mathbf{D}_{-n}^{N} \prod_{i=1}^{N} f(a_i) = \left(\prod_{i=1}^{N} f(a_i) \right) \frac{(-1)^{n-1}}{(2\pi\mathbf{i})^n} \oint \cdots \oint \frac{\sum_{i=1}^{n} \frac{z_n t^{n-i}}{z_i q^{n-i}}}{\left(1 - \frac{tz_2}{qz_1}\right) \cdots \left(1 - \frac{tz_n}{qz_{n-1}}\right)}$$
$$\times \prod_{i<j} \frac{\left(1 - \frac{z_i}{z_j}\right)\left(1 - \frac{qz_i}{tz_j}\right)}{\left(1 - \frac{z_i}{tz_j}\right)\left(1 - \frac{qz_i}{z_j}\right)} \left(\prod_{i=1}^{n} \prod_{i'=1}^{N} \frac{z_i - t^{-1} q a_{i'}}{z_i - q a_{i'}} \right) \prod_{i=1}^{n} \frac{g(z_i) dz_i}{z_i},$$

for any $a_1, \cdots, a_N \in B_r$. The contours are in $B_{r'}$ and nested: all enclose 0 and $qa_{i'}$, and $|z_i| < |tz_{i+1}|$ for each $1 \leq i \leq n-1$.

Apply repeatedly

Apply the operators repeatedly to both sides of the Cauchy identity:

$$\prod_{1\le i\le M_1, 1\le j\le M_2} \frac{\prod_{k=1}^{\infty}(1-ta_ib_jq^{k-1})}{\prod_{k=1}^{\infty}(1-a_ib_jq^{k-1})} = \sum_{\lambda\in\mathbb{Y}} P_\lambda(a_1,\cdots,a_{M_1};q,t)Q_\lambda(b_1,\cdots,b_{M_2};q,t)$$

then one can obtain any mixture moments.

Theorem (Discrete joint moments)

For any positive integers m, n, $\tilde{m}$, $\tilde{n}$, and variables $w_1,\cdots,w_m$, $\tilde{w}_1,\cdots,\tilde{w}_{\tilde{m}}$, denote

$$\mathfrak{I}(w_1,\cdots,w_m;\alpha,M,\theta,n) = \frac{1}{(w_2-w_1+1-\theta)\cdots(w_m-u_{m-1}+1-\theta)} \times \prod_{1\le i<j\le m}\frac{(w_j-w_i)(w_j-w_i+1-\theta)}{(w_j-w_i-\theta)(w_j-w_i+1)}\prod_{i=1}^{m}\frac{w_i-\theta}{w_i+(n-1)\theta}\cdot\frac{w_i-\theta\alpha}{w_i-\theta\alpha-\theta M},$$

Apply repeatedly (cont.)

and

$$\mathfrak{L}(w_1, \cdots, w_m; \tilde{w}_1, \cdots, \tilde{w}_{\tilde{m}}; \theta) = \prod_{1 \le i \le \tilde{m}, 1 \le j \le m} \frac{(\tilde{w}_i - w_j)(\tilde{w}_i - w_j + 1 - \theta)}{(\tilde{w}_i - w_j - \theta)(\tilde{w}_i - w_j + 1)}.$$

Then the expectation of higher moments $\mathfrak{P}_k(x^N)$ can be computed via

$$\mathbb{E}\left(\mathfrak{P}_{k_1}(x^{N_1}) \cdots \mathfrak{P}_{k_l}(x^{N_l})\right) = \frac{(-\theta)^{-l}}{(2\pi \mathbf{i})^{k_1 + \cdots + k_l}} \oint \cdots \oint \prod_{i=1}^{l} \mathfrak{I}(u_{i,1}, \cdots, u_{i,k_i}; \alpha, M, \theta, N_i)$$

$$\times \prod_{i<j} \mathfrak{L}(u_{i,1}, \cdots, u_{i,k_i}; u_{j,1}, \cdots, u_{j,k_j}; \theta) \prod_{i=1}^{l} \prod_{i'=1}^{k_i} du_{i,i'},$$

where for each $i = 1, \cdots, l$, the contours of $u_{i,1}, \cdots, u_{i,k_i}$ enclose $-\theta(N_i - 1)$ but not $\theta(\alpha + M)$, and $|u_{i,1}| \ll \cdots \ll |u_{i,k_i}|$. For $1 \le i < l$, we also require that $|u_{i,k_i}| \ll |u_{i+1,1}|$.

Problem: huge contour integral!

Reduce to one contour

Some cases of the following reduction identity was communicated to the us by Alexei Borodin.

Let s be a positive integer. Let f, $g_1, \cdots, g_s$ be meromorphic functions with possible poles at $\{\mathfrak{p}_1, \cdots, \mathfrak{p}_m\}$. Then for $n \geq 2$,

$$\frac{1}{(2\pi\mathbf{i})^n}\oint\cdots\oint\frac{1}{(v_2-v_1)\cdots(v_n-v_{n-1})}\prod_{i=1}^{n}f(v_i)dv_i\prod_{i=1}^{s}\left(\sum_{j=1}^{n}g_i(v_j)\right)$$
$$=\frac{n^{s-1}}{2\pi\mathbf{i}}\oint f(v)^n\prod_{i=1}^{s}g_i(v)dv,$$

where the contours in both sides are around all of $\{\mathfrak{p}_1, \cdots, \mathfrak{p}_m\}$, and for the left hand side we required $|u_1| \ll \cdots \ll |u_n|$.

An example: in the proof of LLN

From the Theorem of discrete joint moments, there is

$$\mathbb{E}\left(\mathfrak{P}_k(x^N) - \mathfrak{P}_k(x^{N-1})\right) = \frac{(-\theta)^{-1}}{(2\pi\mathbf{i})^k} \oint \cdots \oint$$

$$\times \frac{1}{(u_2 - u_1 + 1 - \theta)\cdots(u_k - u_{k-1} + 1 - \theta)} \prod_{i<j} \frac{(u_j - u_i)(u_j - u_i + 1 - \theta)}{(u_j - u_i + 1)(u_j - u_i - \theta)}$$

$$\times \left(\prod_{i=1}^{k} \frac{u_i - \theta}{u_i + (N-1)\theta} - \prod_{i=1}^{k} \frac{u_i - \theta}{u_i + (N-2)\theta}\right) \prod_{i=1}^{k} \frac{\theta\alpha - u_i}{\theta(\alpha + M) - u_i} du_i.$$

Send $L \to \infty$, setting $u_i \sim L\theta v_i$; note

$$\lim_{L\to\infty} L \left(\prod_{i=1}^{k} \frac{u_i - \theta}{u_i + (N-1)\theta} - \prod_{i=1}^{k} \frac{u_i - \theta}{u_i + (N-2)\theta}\right) = -\prod_{i=1}^{k} \frac{v_i}{v_i + \hat{N}} \left(\sum_{i=1}^{k} \frac{1}{v_i + \hat{N}}\right).$$

An example: in the proof of LLN (cont.)

Thus we have

$$\lim_{L\to\infty} \mathbb{E}\left(\mathfrak{P}_k(x^N) - \mathfrak{P}_k(x^{N-1})\right) = \frac{1}{(2\pi\mathbf{i})^k} \oint \cdots \oint \frac{1}{(v_2 - v_1)\cdots(v_k - v_{k-1})}$$
$$\times \left(\prod_{i=1}^{k} \frac{v_i}{v_i + \hat{N}} \cdot \frac{\hat{\alpha} - v_i}{\hat{\alpha} + \hat{M} - v_i} dv_i\right) \left(\sum_{i=1}^{k} \frac{1}{v_i + \hat{N}}\right).$$

In the dimension reduction identity, take $s = 1$, and

$$f(v) = \frac{v}{v + \hat{N}} \cdot \frac{\hat{\alpha} - v}{\hat{\alpha} + \hat{M} - v}, \quad g_1(v) = \frac{1}{v + \hat{N}},$$

we get

$$\frac{1}{2\pi\mathbf{i}} \oint \left(\frac{v}{v + \hat{N}} \cdot \frac{v - \hat{\alpha}}{v - \hat{\alpha} - \hat{M}}\right)^k \frac{1}{v + \hat{N}} dv.$$

Prove Gaussianity in an alternative form

To prove Gaussianity, we use the following form:

Given a random vector $\mathbf{u} = \{u_i\}_{i=1}^{w} \in \mathbb{R}^w$ such that each moment is finite. If for any $h > 2$, and $v_1, \cdots, v_h \in \{u_1, \cdots, u_w\}$, there is

$$\sum_{\{U_1,\cdots,U_t\}\in\Theta_h} (-1)^{t-1}(t-1)! \prod_{i=1}^{t} \mathbb{E}\left[\prod_{j\in U_i} v_j\right] = 0,$$

then $\mathbf{u}$ is (almost surely) Gaussian.
Here Θ_h be the collection of all unordered partitions of $\{1, \cdots, h\}$:

$$\Theta_h = \left\{\{U_1, \cdots, U_t\} : t \in \mathbb{Z}_+, \bigcup_{i=1}^{t} U_i = \{1, \cdots, h\}, U_i \bigcap U_j = \emptyset, U_i \neq \emptyset\right\}.$$

The proof is via the moment generating function, and the above just says that all cumulants of order ≥ 3 vanishes. By an induction argument it is also equivalent to Wick's formula.

What we have here: a Gaussian type asymptote

Let $k_1, \cdots, k_h$ and $N_1 \leq \cdots \leq N_h$ be positive integers, and let $D \subset \{1, \cdots, h\}$ be a subset of indexes, such that for any $1 \leq i < j \leq h$, and $j \in D$, $N_i < N_j$. For any $i \in D$, denote

$$\mathfrak{E}_i = \mathfrak{P}_{k_i}(x^{N_i}) - \mathfrak{P}_{k_i}(x^{N_i - 1}) - \mathbb{E}\left(\mathfrak{P}_{k_i}(x^{N_i}) - \mathfrak{P}_{k_i}(x^{N_i - 1})\right),$$

and for any $i \notin D$, denote

$$\mathfrak{E}_i = \mathfrak{P}_{k_i}(x^{N_i}) - \mathbb{E}\left(\mathfrak{P}_{k_i}(x^{N_i})\right).$$

Then

$$\lim_{L \to \infty} L^{\eta} \sum_{\{U_1, \cdots, U_t\} \in \Theta_h} (-1)^{t-1}(t-1)! \prod_{i=1}^{t} \mathbb{E}\left[\prod_{j \in U_i} \mathfrak{E}_j\right] = 0,$$

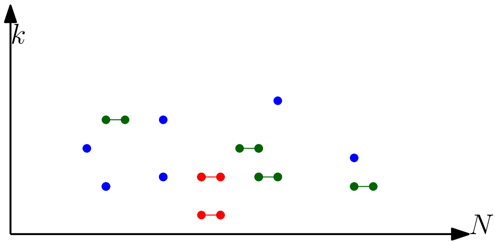
for any $\eta < h - 2 + |D|$ (for most cases this is more than needed).

Proof ideas

- Expand the multiplication to a summation of mixed moments.
- Write as a summation of contour integrals, by the Theorem of discrete joint moments.
- The requirement "$i \neq j \in D$ for $N_i \neq N_j$" ensures that the order of contour integral is unchanged; the contours of $\mathfrak{P}_{k_i}(x^{N_i-1})$ or $\mathfrak{P}_{k_i}(x^{N_i})$ is always inside the contours of $\mathfrak{P}_{k_j}(x^{N_j-1})$ or $\mathfrak{P}_{k_j}(x^{N_j})$, for $i < j$.
- Exploit cancellations: combinatoric identities / graph model.
- Analyze order of decay for the remaining terms.

The Gaussian type asymptote is not actual Gaussian

One cannot take differences of the same level (for $i \neq j \in D$, $N_i \neq N_j$).



Indeed, the covariance

$$\mathbb{E}\left[\prod_{i=1}^{2}\left(\mathfrak{P}_{k_i}(x^N) - \mathfrak{P}_{k_i}(x^{N-1}) - \mathbb{E}\left(\mathfrak{P}_{k_i}(x^N) - \mathfrak{P}_{k_i}(x^{N-1})\right)\right)\right]$$

only decays at the order of L^{-1}.

Passing to actual Gaussianity: discrete levels

Same formula as the Gaussian type asymptote, but need to remove the "different level" condition.

- Write as a summation of $2^{|D|-1}$ expressions, each satisfying the "different level" condition, e.g.

$$\mathbb{E}\left[\prod_{i=1}^{2}\left(\mathfrak{P}_{k_i}(x^N) - \mathfrak{P}_{k_i}(x^{N-1}) - \mathbb{E}\left(\mathfrak{P}_{k_i}(x^N) - \mathfrak{P}_{k_i}(x^{N-1})\right)\right)\right]$$

$$= \mathbb{E}\left[\left(\mathfrak{P}_{k_1}(x^N) - \mathfrak{P}_{k_1}(x^{N-1}) - \mathbb{E}\left(\mathfrak{P}_{k_1}(x^N) - \mathfrak{P}_{k_1}(x^{N-1})\right)\right) \times \left(\mathfrak{P}_{k_2}(x^N) - \mathbb{E}\mathfrak{P}_{k_2}(x^N)\right)\right]$$

$$- \mathbb{E}\left[\left(\mathfrak{P}_{k_1}(x^N) - \mathfrak{P}_{k_1}(x^{N-1}) - \mathbb{E}\left(\mathfrak{P}_{k_1}(x^N) - \mathfrak{P}_{k_1}(x^{N-1})\right)\right) \times \left(\mathfrak{P}_{k_2}(x^{N-1}) - \mathbb{E}\mathfrak{P}_{k_2}(x^{N-1})\right)\right]$$

- This is at the price of slower decay, but still enough.

Passing to actual Gaussianity: integral cross levels

For the Gaussianity of integral in y-direction, we need

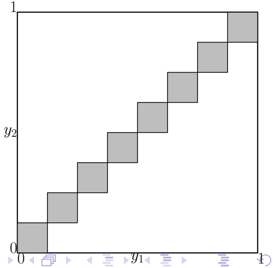
$$\lim_{L\to\infty} L^h \sum_{\{U_1,\cdots,U_t\}\in\Theta_h} (-1)^{t-1}(t-1)! \prod_{i=1}^{t} \mathbb{E}\left[\prod_{j\in U_i}\int_0^1 g_j(y_j)\mathfrak{C}_j(Ly_j)dy_j\right] = 0,$$

where

$$\mathfrak{C}_i(y) = \mathfrak{P}_{k_i}(x^{\lfloor y\rfloor}) - \mathfrak{P}_{k_i}(x^{\lfloor y\rfloor-1}) - \mathbb{E}\left(\mathfrak{P}_{k_i}(x^{\lfloor y\rfloor}) - \mathfrak{P}_{k_i}(x^{\lfloor y\rfloor-1})\right).$$

Need to be careful, since when writing as sum of contour integrals, the order of contours changes when the order of $y_1,\cdots,y_h$ changes.

Do integral for each area separately.



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